AVERAGE CASE COMPLEXITY OF LINEAR MULTIVARIATE PROBLEMS PART II: APPLICATIONS

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Abstract. We apply the theoretical results of Part I to linear multivariate problems LMP equipped with the folded Wiener sheet measure. We are particularly interested in multivariate weighted integration and multivariate function approximation. We prove that any LMP which satisfies (A.1) of Part I is tractable and its exponent is at most 2. We show that optimal or nearly optimal sample points can be derived from hyperbolic cross points, and exhibit nearly optimal algorithms. In particular, we find the order of the average case complexity of multivariate function approximation in Λ^{std}.

1. Introduction

In this part we study linear multivariate problems LMP which are equipped with a particular Gaussian measure which is called the folded Wiener sheet measure. Of particular interest are multivariate weighted integration and multivariate function approximation.

Using the theoretical results of Part I, we establish tractability of any linear multivariate problem LMP which satisfies (A.1) of Part I and show that its exponent is at most 2 in

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both $\Lambda^{\rm all}$ and $\Lambda^{\rm std}$. Furthermore, for multivariate weighted integration, the exponent is at most 1 in $\Lambda^{\rm std}$ (obviously it is 0 in $\Lambda^{\rm all}$).

For multivariate function approximation the exponents in $\Lambda^{\rm std}$ and $\Lambda^{\rm all}$ are the same. The exponent in $\Lambda^{\rm all}$ was already known due to Papageorgiou and Wasilkowski [90]. In $\Lambda^{\rm std}$, it was known that the exponent is at most 6, see Papageorgiou [90].

Tractability of multivariate function approximation in the average case setting is significant since it is known that the randomized case complexity is of the same order as the worst case complexity, see Wasilkowski [89]. Thus, unlike multivariate weighted integration, intractability of multivariate function approximation in the worst case setting cannot be broken by the randomized setting. (Multivariate function approximation is intractable in the worst case setting iff at least one of the r_i is zero; the r_i are defined later.)

Before we turn to construction of optimal or nearly optimal sample points in the average case setting we briefly survey known results for multivariate integration and multivariate function approximation. (We are not aware of average case analysis of other multivariate problems.)

The problem of multivariate integration has been extensively studied in the worst case setting, see books and surveys of Niederreiter [78, 88], Novak [88] and Traub, Wasilkowski and Woźniakowski [88] for hundreds of references. For our study the most relevant papers from the worst case setting are Bykovskij [85], Frolov [76] and Temlyakov [90].

The average case setting for multivariate integration INT_d has been studied in a relatively few papers. For the folded Wiener sheet measure, optimal design and average case complexity are known for the scalar case $d=1$, see Lee and Wasilkowski [86], and Sacks and Ylvisaker [70]. The average case complexity is

$$\mathrm{comp}^{\rm avg}(\varepsilon;\mathrm{INT}_1) \;=\; \Theta\left(c\,\left(\frac{1}{\varepsilon}\right)^{1/(r+1)}\right),$$

where, as in Part I, c denotes the cost of one function evaluation, and r denotes the smoothness of scalar functions.

For the multivariate case, $d>1$, let r_i denote how often we can differentiate a function with respect to the ith variable. Let $r_{\min}=\min r_i$. Papageorgiou and Wasilkowski [90], see also Ylvisaker [75], showed that grid points are a poor choice of sample points. More precisely, to achieve the average error ε we have to take n grid points with

$$n \;=\; \Theta\left(\left(\frac{1}{\varepsilon}\right)^{d/(r_{\min}+1)}\right).$$

For fixed $r_{\min}$, the number of grid points depends *exponentially* on d. Since the exponent of INT is 1, $n = O(\varepsilon^{-1-\delta})$, for any $\delta > 0$, sample points are sufficient to achieve the average error ε, but how to construct these points has been a challenging problem.

Recently, the optimal design problem has been solved modulo a multiplicative factor which may depend on d. For the classical Wiener measure, i.e., $r_i = 0, \forall i \le d$, Woźniakowski [91] showed that this problem is related to minimal L_2 discrepancy. The latter problem was solved by Roth [54, 80], see also Frolov [80] for a sharp upper bound on discrepancy. In this way optimal sample points were derived from Hammersley points. For the folded Wiener sheet measure, Paskov [91] utilized a relation between the average case and worst case settings and derived optimal sample points by using the worst case analysis of Bykovskij [85], Frolov [76] and Temlyakov [90].

We now turn to multivariate function approximation APP_d. This problem has been studied for arbitrary Gaussian measures and Λ^{all} by Papageorgiou and Wasilkowski [90], for the folded Wiener measure for $d = 1$ and Λ^{std} by Speckman [76], and by Papageorgiou and Wasilkowski [90] for $d \ge 1$, and for the classical Wiener measure and Λ^{std} by Lee [86] for $d = 1$.

For the scalar case $d = 1$, Speckman [76] showed that equally spaced sample points are optimal modulo a multiplicative factor. From this and Papageorgiou and Wasilkowski [90], we know that the average case complexity functions in both Λ^{std} and Λ^{all} differ by at most a multiplicative factor and are given by

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon; \mathrm{APP}_1) \;=\; \Theta\left(c \left(\frac{1}{\varepsilon}\right)^{1/(r+1/2)} \right).$$

For the classical Wiener measure, the multiplicative factors in the Θ notation are known for Λ^{std} and Λ^{all} and the ratio of the average case complexity functions in these classes is roughly 1.645, see Lee [86], and Papageorgiou and Wasilkowski [90].

For the multivariate case $d \ge 1$, Papageorgiou and Wasilkowski [90] showed that if n grid points are used, then to guarantee an average error ε, we must take

$$n \;=\; \Theta\left(\left(\frac{1}{\varepsilon}\right)^{d/(r_{\min}+1/2)} \right).$$

Thus, for fixed $r_{\min}$ the number of grid points is *exponential* in d. Hence, as for multivariate integration, grid points are a poor choice of sample points.

Papageorgiou and Wasilkowski [90] *conjectured* that there exist n sample points which lead to the average error

$$e_n^{\mathrm{avg}} \;=\; \Theta\left(\frac{(\log n)^{(k^*-1)(r_{\min}+1)}}{n^{r_{\min}+1/2}} \right)$$

or equivalently that the average case complexity is

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon;\mathrm{APP}_d) \;=\; \Theta\left(c\,\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1/2)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(r_{\min}+1)/(r_{\min}+1/2)}\right), \qquad (1.1)$$

where k^* denotes the number of variables with the minimal regularity $r_{\min}$.

We now turn to construction of optimal or nearly optimal sample points for linear multivariate problems equipped with the folded Wiener sheet measure μ. The basic properties of μ are surveyed in Section 2. Here, we only mention that μ is the classical Wiener measure placed on partial derivatives $(r_1, r_2, \dots, r_d)$ of functions of d variables.

As in Part I, the problem of optimal sample points in the average case setting is solved by utilizing relations to the worst case setting for linear multivariate problems in the reproducing kernel Hilbert space H_μ. In our case, H_μ is a Sobolev space of smooth *nonperiodic* functions which are differentiable $(r_1+1, r_2+1, \dots, r_d+1)$ times and satisfy certain boundary conditions.

Multivariate integration in the worst case setting has been studied in this Sobolev space by Bykovskij [85], Frolov [76] and Temlyakov [90]. Using the worst case results of these papers, Paskov [91] found the average case complexity,

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon;\mathrm{INT}_d) \;=\; \Theta\left(c\,\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)/(2+2r_{\min})}\right). \qquad (1.2)$$

Multivariate function approximation in the worst case setting has been studied in this Sobolev space additionally assuming *periodicity* of functions by Temlyakov [87, 91]. He introduced *hyperbolic cross* points and proved that they are nearly optimal sample points. Hyperbolic cross points are defined as a subset of grid points whose indices satisfy a "hyperbolic" inequality. Approximation of periodic functions by trigonometric polynomials that use Fourier coefficients with these hyperbolic cross indices was initiated by Babenko [60].

It is easy to extend Temlyakov's result to the nonperiodic case. We show that, modulo a multiplicative factor which may depend on d, optimal sample points for multivariate function approximation APP_d in the average case setting are derived from hyperbolic cross points, and the average case complexity is given by

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon;\mathrm{APP}_d) \;=\; \Theta\left(c\,\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1/2)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(r_{\min}+1)/(r_{\min}+1/2)}\right).$$

This proves the conjecture of (1.1). The upper bound is achieved by computing a linear combination of function values at sample points derived from hyperbolic cross points.

For simplicity, in this paper we address only the construction of sample points for multivariate weighted integration $\rho\mathrm{INT}_d$, see Section 5, and multivariate function approximation APP_d, see Section 6. In both cases, points derived from hyperbolic cross points are nearly optimal. In particular, for multivariate integration, i.e., $\rho \equiv 1$, points derived from hyperbolic cross points lead to a slightly larger power of $\log 1/\varepsilon$ in (1.2). For specific estimates we refer to Sections 5 and 6.

For general linear multivariate problems $\mathrm{LMP} = \{\mathrm{LMP}_d\}$ we may apply one of the two approaches of Part I. That is, we may treat LMP_d as a number of multivariate weighted integration problems and apply the construction of Section 5, or we may bound LMP_d by a multiple of multivariate function approximation APP_d and apply the construction of Section 6. In this way, points derived from hyperbolic sample points may be used for general linear multivariate problems.

2. Folded Wiener Sheet Measures

In this section we summarize the main properties of the folded Wiener sheet measures, see Papageorgiou and Wasilkowski [90], which will be used in this paper.

Let $D = [0,1]^d$ be the d dimensional unit cube. Let $C^{\vec{r}} = C^{r_1,r_2,\ldots,r_d}(D)$ be the class of functions f, $f : D \to \mathbb{R}$, which are r_j times continuously differentiable with respect to t_j, $j = 1,2,\ldots,d$, where $\vec{r} = [r_1,r_2,\ldots,r_d]$ and r_j are nonnegative integers. Let $D^{\vec{i}} = D^{i_1,i_2,\ldots,i_d}$ denote the derivative operator,

$$D^{\vec{i}} f \;=\; \frac{\partial^{i_1+i_2+\cdots+i_d}}{\partial x_1^{i_1}\,\partial x_2^{i_2}\ldots\partial x_d^{i_d}}\; f, \quad \forall f \in C^{\vec{r}},$$

for $i_j \le r_j$, $j = 1,2,\ldots,d$.

Consider the Banach space F given by

$$F \;=\; \{f \in C^{\vec{r}} :\; (D^{i_1,i_2,\ldots,i_d} f)(t) = 0,\ \forall i_j \le r_j,\ j = 1,2,\ldots,d, \qquad \text{when one of the components of } t \text{ is zero}\} \tag{2.1}$$

with the sup norm $\|f\| = \sup_{t\in D} |D^{r_1,r_2,\ldots,r_d} f(t)|$. Clearly, F is a subset of $L_2(D)$ as assumed in Part I.

The space F is equipped with the folded Wiener sheet measure,

$$\mu(B) \;=\; w(D^{\vec{r}} B), \quad \text{for any Borel set } B \text{ of } F, \tag{2.2}$$

where w is the classical Wiener sheet measure defined on the space $C = C^{\vec{0}}(D)$ of continuous functions, see e.g., Kuo [75] and Adler [81]. That is, w is Gaussian with mean zero and covariance kernel

$$R_w(t,x) \;=\; \prod_{j=1}^d \min\{t_j, x_j\}.$$

Note that for $\vec{r} = \vec{0}$ we get $\mu = w$.

The covariance kernel R_μ of the measure μ is given by

$$R_\mu(t,x) \;=\; \prod_{j=1}^d \int_0^1 \frac{(t_j - s)_+^{r_j}}{r_j!}\, \frac{(x_j - s)_+^{r_j}}{r_j!}\, ds.$$

Observe that

$$R_\mu(t,t) \;=\; \frac{\prod_{j=1}^d t_j^{2r_j+1}}{\prod_{j=1}^d (2r_j+1)(r_j!)^2} \;\le\; K_2 \;=\; \frac{1}{\prod_{j=1}^d (2r_j+1)(r_j!)^2} \;\le\; 1.$$

This means that the assumption (A.2) of Part I holds with K_2 which is at most 1, and $K_2 = 1$ iff $\vec{r} = \vec{0}$.

The folded Wiener sheet measure μ of (2.2) generates the reproducing kernel Hilbert space H_μ, see Section 4.1 of Part I and Paskov [91]. The space H_μ,

$$H_\mu = W_2^{\overline{r+1}} = W_2^{r_1+1,\dots,r_d+1}(D), \tag{2.3}$$

is now the space of functions f of the form

$$f(x) \;=\; \int_D \prod_{j=1}^d \frac{(x_j - t_j)_+^{r_j}}{r_j!}\, \phi(t_1, t_2, \dots, t_d)\, dt_1\, dt_2 \dots dt_d, \quad \forall x \in D, \tag{2.4}$$

where $\phi \in L_2(D)$, see Temlyakov [90].

Observe that a function f from H_μ is (r_j+1) times differentiable with respect to x_j and $D^{r_1+1,\dots,r_d+1} f = \phi \in L_2(D)$. We also have $D^{i_1,\dots,i_d} f(t) = 0$ for all $i_j \le r_j$ and all t with at least one component equal to zero.

The inner product of H_μ is given by

$$\langle f, g\rangle_\mu \;=\; \int_D D^{r_1+1,\dots,r_d+1} f(t)\, D^{r_1+1,\dots,r_d+1} g(t)\, dt.$$

Let $I_d : F \to L_2(D)$ be the embedding operator from the space F of (2.1) to $L_2(D)$. Let $\nu = \mu\, I_d^{-1}$ be the measure on the Hilbert space $L_2(D)$. Then ν is Gaussian with mean zero.

Let C_ν be the covariance operator of ν. Papageorgiou and Wasilkowski [90] show that the eigenfunctions η_j and eigenvalues λ_j of the covariance operator C_ν are the solutions z and λ of the differential equation

$$\lambda\, D^{2r_1+2,\dots,2r_d+2}\, z \;-\; (-1)^{d+r_1+\cdots+r_d}\, z \;=\; 0$$

with boundary conditions $\partial^i z(x)/\partial x_j^i = 0$ for $x_j = 0$ and $i = 0, 1, \dots, r_j$, and $\partial^i z(x)/\partial x_j^i = 0$ for $x_j = 1$ and $i = r_j + 1, \dots, 2r_j + 1$, where $j = 1, 2, \dots, d$.

Let

$$r_{\min} \;=\; \min\{r_j \,:\, 1 \le j \le d\} \tag{2.5}$$

denote the minimal regularity in the space F. Let

$$k^* \;=\; \mathrm{card}\big(\{j \,:\; r_j = r_{\min}\}\big) \tag{2.6}$$

denote the number of directions with minimal regularity. Clearly, $1 \le k^* \le d$. The ordered eigenvalues of C_ν are given by

$$\begin{aligned} \lambda_j \;&=\; \Theta\left(\left(\frac{(\log j)^{k^*-1}}{j}\right)^{2r_{\min}+2}\right), \\ \sqrt{\sum_{j\ge k+1} \lambda_j} \;&=\; \Theta\left(\frac{(\log n)^{(k^*-1)(r_{\min}+1)}}{n^{r_{\min}+1/2}}\right). \end{aligned} \tag{2.7}$$

3. Worst Case Estimates

As shown in Part I, linear multivariate problems in the average case setting are related to multivariate function approximation problems in the worst case setting. In this section we study the worst case setting of the following two multivariate function approximation problems,

$$\begin{aligned} \mathrm{APP}_d^{\mathrm{wor},2} \;&= \{BH_\mu, L_2(D), I_d, \Lambda^{\mathrm{std}}\}, \\ \mathrm{APP}_d^{\mathrm{wor},\infty} \;&= \{BH_\mu, L_\infty(D), I_d, \Lambda^{\mathrm{std}}\} \end{aligned}$$

In our case, the reproducing kernel Hilbert space $H_\mu = W_2^{\overline{r+1}}$ is given by (2.3) and (2.4).

Let $W_0^{\overline{r+1}}$ be a subspace of $H_\mu = W_2^{\overline{r+1}}$ of periodic functions for which $D^{i_1,\dots,i_d} f\,(t) = 0$ for all $i_j \le r_j$ and all t with at least one component equal to zero or one, i.e., t belongs to the boundary of D. Obviously, the unit ball $BW_0^{\overline{r+1}}$ is a proper subset of the unit ball $BW_2^{\overline{r+1}}$.

Multivariate function approximation for the unit ball $BW_0^{\overline{r+1}}$ of periodic functions in the worst case setting has been analyzed by Temlyakov [87]. We now present Temlyakov's construction of nearly optimal sample points and nearly optimal linear U.

Assume first that $d = 1$. Let

$$V_m(x) \;=\; 1 + 2\sum_{j=1}^{m} \cos jx \;+\; 2\sum_{j=m+1}^{2m} \frac{2m-j}{m}\cos jkx$$

be the *de la Vallée-Poussin kernel* of order $2m-1$. Define

$$I_m(f,x) \;=\; \frac{1}{8m}\sum_{i=1}^{8m-1} V_m\left(2\pi\left(x-\frac{i}{8m}\right)\right) f\left(\frac{i}{8m}\right), \quad \forall x\in[0,1].$$

Observe that I_m uses function values at sample points $i/(8m)$. Let

$$\Delta_0(f,x) \;=\; I_1(f,x),$$
$$\Delta_k(f,x) \;=\; I_{2^k}(f,x) \;-\; I_{2^{k-1}}(f,x), \quad k\ge 1.$$

For $d\ge 1$ we proceed as follows. For $\vec{s} = [s_1, s_2, \dots, s_d]$ and $x = [x_1, x_2, \dots, x_d]$, let

$$\Delta_{\vec{s}}(f,x) \;=\; \Delta_{s_1}(\,\Delta_{s_2}\cdots\Delta_{s_d}(f,x_d)\cdots),$$

where Δ_{s_j} acts as a one dimensional operator on a function depending on the variable x_j.

Without loss of generality, we may assume that

$$\vec{r} \;=\; [r_{\min},\dots,r_{\min}, r_{k^*+1},\dots,r_d],$$

where $r_j > r_{\min}$ for $j > k^*$. Define the vector $\vec{r'} = [r_1', r_2',\dots,r_d']$ such that

$$r_j' \;=\; r_{\min} \qquad \text{for } j = 1,2,\dots,k^*.$$
$$r_j' \;\in\; (r_{\min}, r_j) \qquad \text{for } j = k^*+1,\dots,d.$$

In what follows, by $\overline{r'+1}$ we denote the vector $[r_1'+1,\dots,r_d'+1]$ and by $(\overline{r'+1},\vec{s})$ we mean $\sum_{j=1}^d (r_j'+1)s_j$.

For given integer m, consider the *hyperbolic cross* X_n of sample points,

$$X_n \;=\; \left\{\left(\frac{l_1}{2^{s_1+3}},\frac{l_2}{2^{s_2+3}},\dots,\frac{l_d}{2^{s_d+3}}\right) : \, l_j = 1,2,\dots,2^{s_j+3}-1,\; j=1,2,\dots,d,\;\vec{s}\in A\right\}, \tag{3.1}$$

where A is the set of all nonnegative integer vectors $\vec{s}$ for which $(\overline{r'+1}, \vec{s}) \le m\,(r_{\min}+1)$. Here $n = \mathrm{card}(X_n)$,

$$n \;=\; \Theta\left(2^m\, m^{k^*-1}\right).$$

Define

$$T_n(f,x) \;=\; \sum_{(\overline{r'+1},\vec{s}) \,\le\, m(r_{\min}+1)} \Delta_{\vec{s}}(f,x). \tag{3.2}$$

The operator T_n uses function values $f(x_j)$ at sample points x_j from the hyperbolic cross X_n. It depends linearly on $f(x_j)$ and can be written as

$$T_n(f,x) \;=\; \sum_{x_j \in X_n} f(x_j)\, a_j(x), \tag{3.3}$$

where a_j is a linear combination of the de la Vallée-Poussin kernel.

Temlyakov proved that for periodic f from the unit ball $BW_0^{\overline{r+1}}$ we have

$$\begin{aligned} \|f - T_n(f)\|_{L_2(D)} \;&=\; O\left(\frac{1}{n^{r_{\min}+1}}\,(\log n)^{(k^*-1)(r_{\min}+2)}\right), \\ \|f - T_n(f)\|_{L_\infty(D)} \;&=\; O\left(\frac{1}{n^{r_{\min}+1/2}}\,(\log n)^{(k^*-1)(r_{\min}+1)}\right), \end{aligned} \tag{3.4}$$

see Temlyakov [87] for the estimate in $L_2(D)$ and Temlyakov [91] for $L_\infty(D)$.

We need to extend Temlyakov's result for nonperiodic functions from the unit ball $BW_2^{\overline{r+1}}$. This can be done in many ways. Here we opt for simplicity at the expense of a possibly larger constant. Define

$$\begin{aligned} h(t) \;&=\; 4\,t\,(1-t), \qquad \forall t \in [0,1], \\ \vec{h}(x) \;&=\; \big(h(x_1), h(x_2), \dots, h(x_d)\big) \qquad \forall x \in D. \end{aligned} \tag{3.5}$$

Then $h([0,1/2]) = h([0,1]) = [0,1]$. Thus, $\vec{h}([0,1/2]^d) = D$. For $f \in BW_2^{\overline{r+1}}$ define

$$g(x) \;=\; f\big(\vec{h}(x)\big), \qquad \forall x \in D.$$

Observe that g is periodic and enjoys the same smoothness as f. That is, $g \in W_0^{\overline{r+1}}$. There exists a constant $K = K(d, \vec{r})$ such that

$$\|g\|_\mu \;\le\; K.$$

Indeed, $D^{r_1+1,\dots,r_d+1} g$ consists of the terms

$$a(x) \;=\; \alpha\, f^{(i_1,\dots,i_d)}\big(h(x_1),\dots,h(x_d)\big)\,\big(h'(x_1)\big)^{k_1} \cdots \big(h'(x_d)\big)^{k_d},$$

where $\alpha = \alpha_{i_1,\dots,i_d}$ is a constant, and $i_j,\, k_j \le r_j + 1$. Furthermore, if $i_j = r_j + 1$ then $k_j = r_j + 1$.

The functions $a(\cdot)$ are uniformly bounded in $L_2(D)$. To show this assume for simplicity that $i_j = r_j + 1$ for $j = 1, 2, \dots, k$. Then we change variables by setting $t_j = h(x_j)$, $j = 1, 2, \dots, k$, in all subcubes $D_1 \times \cdots \times D_d$ with $D_i = [0, 1/2]$ or $D_i = [1/2, 1]$. In each such subcube h is one-to-one. This way we reduce the function $a(\cdot)$ to the form

$$\alpha\, f^{(r_1+1,\dots,r_k+1,i_{k+1},\dots,i_d)}\big(t_1,\dots,t_k,h(x_{k+1}),\dots,h(x_d)\big) \prod_{j=1}^{k} \big(h'(x_j)\big)^{r_j} \prod_{j=k+1}^{d} \big(h'(x_j)\big)^{k_j}.$$

Observe that $i_j < r_j + 1$, $j = k+1, \dots, d$, and (2.4) imply

$$\int_{[0,1]^k} \left(f^{(r_1+1,\dots,r_k+1,i_{k+1},\dots,i_d)}\big(t_1,\dots,t_k,h(x_{k+1}),\dots,h(x_d)\big) \right)^2 dt_1 \cdots dt_k \;\le\; 1.$$

Since $|h'(x)|$ is bounded by 4, $\|a\|_d = O(1)$, and $\|g\|_\mu \le K$ for some K, as claimed.

Thus, we may apply (3.2) to the function $g(\cdot)/K$ and (3.4) yields

$$\begin{aligned} \|g - T_n(g)\|_{L_2(D)} &= O\left(\frac{1}{n^{r_{\min}+1}}\, (\log n)^{(k^*-1)(r_{\min}+2)} \right), \\ \|g - T_n(g)\|_{L_\infty(D)} &= O\left(\frac{1}{n^{r_{\min}+1/2}}\, (\log n)^{(k^*-1)(r_{\min}+1)} \right). \end{aligned} \tag{3.6}$$

Define

$$T_n^*(f,t) \;=\; T_n\left(g, \vec{h}^{-1}(t)\right), \tag{3.7}$$

where $\vec{h}^{-1}(t) = \big(\frac{1}{2}(1-\sqrt{1-t_1}), \dots, \frac{1}{2}(1-\sqrt{1-t_d})\big)$, $t \in D$. From (3.3) we have

$$T_n^*(f,t) \;=\; \sum_{x_j \in X_n} f\big(\vec{h}(x_j)\big)\, a_j\big(\vec{h}^{-1}(t)\big) \;=\; \sum_{j=1}^{n} f(x_n^*) h_j^*(t), \tag{3.8}$$

where the points x_j^* are derived from the hyperbolic cross points x_j, i.e,

$$x_j^* = \vec{h}(x_j),$$

and $h_j^*(t) = a_j\big(\vec{h}^{-1}(t)\big)$.

We now estimate the error between f and $T_n^*(f)$ in $L_2(D)$. Observe that

$$\|g - T_n(g)\|_d^2 \;\ge\; \int_{[0,1/2]^d} \Big(f\big(\vec{h}(x)\big) \;-\; T_n\big(f(\vec{h}(\cdot), x\big) \Big)^2 dx.$$

The function $\vec{h}$ is one-to-one on $[0,1/2]^d$. Changing variables $t_j = h(x_j)$ we get $t_j \in [0,1]$ and

$$\|g - T_n(g)\|_d^2 \geq \int_D (f(t) - T_n^*(f,t))^2 \, \frac{1}{h'(x_1)\cdots h'(x_d)}\, dt.$$

Since $\left(h'(x)\right)^{-1} \geq \frac{1}{4}$, we finally get

$$\|g - T_n(g)\|_d = \Omega\left(\|f - T_n^*(f)\|_d\right).$$

It is also easy to estimate the error between f and $T_n^*(f)$ in $L_\infty(D)$. Indeed, we have

$$\|g - T_n(g)\|_{L_\infty(D)} \geq \|f(\vec{h}) - T_n(f(\vec{h}))\|_{L_\infty([0,1/2]^d)} \geq \|f - T_n^*(f)\|_{L_\infty(D)}.$$

From (3.6) we thus conclude that for all $f \in B\overline{W_2^{r+1}}$ we have

$$\begin{aligned}
\|f - T_n^*(f)\|_{L_2(D)} &= O\left(\frac{1}{n^{r_{\min}+1}} \, (\log n)^{(k^*-1)(r_{\min}+2)}\right),\\
\|f - T_n^*(f)\|_{L_\infty(D)} &= O\left(\frac{1}{n^{r_{\min}+1/2}} \, (\log n)^{(k^*-1)(r_{\min}+1)}\right).
\end{aligned} \tag{3.9}$$

Thus, modulo a multiplicative factor which may depend on d, we can approximate nonperiodic functions as well as periodic ones.

It is known that the first estimate of (3.4) is optimal modulo a power of $\log n$, and the second estimate of (3.4) is optimal modulo a a multiplicative factor which may depend on d. Hence, the estimates (3.9) are also optimal. In particular, as in Part I, let $e_n^{\text{wor}}(\text{APP}_d^{\text{wor},2})$ and $e_n^{\text{wor}}(\text{APP}_d^{\text{wor},\infty})$ be the minimal worst case errors which can be achieved by computing n sample points. Then

$$\begin{aligned}
e_n^{\text{wor}}(\text{APP}_d^{\text{wor},2}) &= \Theta\left(\frac{1}{n^{r_{\min}+1}} \, (\log n)^{(k^*-1)(r_{\min}+1+\alpha(n))}\right), \ \alpha(n) \in [0,1],\\
e_n^{\text{wor}}(\text{APP}_d^{\text{wor},\infty}) &= \Theta\left(\frac{1}{n^{r_{\min}+1/2}} \, (\log n)^{(k^*-1)(r_{\min}+1)}\right).
\end{aligned} \tag{3.10}$$

This means that the sample points $x_j^* = \vec{h}(x_j)$ with hyperbolic cross points x_j and the linear T_n^* are nearly optimal for $\text{APP}_d^{\text{wor},2}$ and $\text{APP}_d^{\text{wor},\infty}$ in the worst case setting. In particular, from (3.10) we obtain the estimates of the worst case complexity functions of $\text{APP}_d^{\text{wor},2}$ and $\text{APP}_d^{\text{wor},\infty}$,

$$\begin{aligned}
\text{comp}^{\text{wor}}(\varepsilon;\text{APP}_d^{\text{wor},2}) &= \Theta\left(c\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(1+\alpha(\varepsilon))}\right), \ \alpha(\varepsilon) \in [0,\tfrac{1}{r_{\min}+1}],\\
\text{comp}^{\text{wor}}(\varepsilon;\text{APP}_d^{\text{wor},\infty}) &= \Theta\left(c\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1/2)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(r_{\min}+1)/(r_{\min}+1/2)}\right).
\end{aligned}$$

4. Tractability

We study the tractability of a linear multivariate problem LMP $= \{\mathrm{LMP}_d\}$, where

$$\mathrm{LMP}_d \;=\; \{F, \mu, G, S, \Lambda^{\mathrm{std}}\},$$

with F and μ given by (2.1) and (2.2).

We assume that S satisfies (A.1) of Part I. Since $K_2 \le 1$, Theorem 3.3 of Part I states that tractability of LMP follows from tractability of the multivariate function approximation problem APP $= \{\mathrm{APP}_d\}$ with

$$\mathrm{APP}_d \;=\; \{F, \mu, L_2(D), I_d, \Lambda^{\mathrm{std}}\}.$$

Hence, it is enough to check tractability of APP. Due to Theorem 3.1 and Corollary 3.1 of Part I, tractability of APP is equivalent to tractability in Λ^{all}. This follows from Papageorgiou and Wasilkowski [90]. They proved that the average case complexity of APP_d in Λ^{all} is given by

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon; \mathrm{APP}_d) \;=\; \Theta\left(c \left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1/2)} \left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(r_{\min}+1)/(r_{\min}+1/2)}\right), \qquad (4.1)$$

where $r_{\min}$ and k^* are defined in (2.5) and (2.6).

Since $1/(r_{\min}+1/2) \le 2$, multivariate function approximation APP is tractable in Λ^{all}. To find the exponent of APP, note that the components of the vector $\vec{r} = [r_1, r_2, \dots, r_d]$ may depend on d, $r_i = r_i(d)$. Therefore $r_{\min} = r_{\min}(d)$ may also be a function of d. From (4.1) we conclude that the exponent of APP in Λ^{all} is given by

$$p^*(\Lambda^{\mathrm{all}}) \;=\; \frac{1}{1/2 \,+\, \min\{r_j(d):\; j = 1, 2, \dots, d \text{ and } d = 1, 2, \dots\}} \;\le\; 2. \qquad (4.2)$$

We add in passing that (4.2) also follows from Theorem 3.2 of Part I and (2.7).

Since (A.1) and (A.2) are satisfied, APP is also tractable in Λ^{std} with the exponent

$$p^*(\Lambda^{\mathrm{std}}) \;\le\; p^*(\Lambda^{\mathrm{all}}) \,+\, 2 \;\le\; 4.$$

In fact, we show in Section 6 that $p^*(\Lambda^{\mathrm{std}}) = p^*(\Lambda^{\mathrm{all}})$ and that the average case complexity functions of APP_d in Λ^{std} and Λ^{all} may differ only by a multiplicative factor depending on d. We summarize the discussion of this section in the following theorem.

Theorem 4.1.

Any linear multivariate problem which satisfies (A.1) of Part I and which is equipped with the folded Wiener sheet measure is tractable and has exponent at most 2.

5. Multivariate Weighted Integration

As in Section 4.1 of Part I, we consider multivariate weighted integration

$$\rho\mathrm{INT} = \{\rho\mathrm{INT}_d\} \quad \text{with} \quad \rho\mathrm{INT}_d \,=\, \{F, \mu, \mathbb{R}, S, \Lambda^{\mathrm{std}}\},$$

where F and μ are given by (2.1) and (2.2), and

$$S(f) \,=\, \int_D \rho(t)\, f(t)\, dt, \qquad \forall f \in F.$$

For simplicity we assume that the weight ρ is normalized, $\|\rho\|_d = 1$.

Corollaries 4.3 and 4.4 of Part I state relations between multivariate weighted integration $\rho\mathrm{INT}_d$ in the average case setting and multivariate function approximation $\mathrm{APP}_d^{\mathrm{wor},2}$ in the worst case setting. Since $e_n^{\mathrm{wor}}(\mathrm{APP}_d^{\mathrm{wor},2})$ is given by (3.10), (ii) of Corollary 4.3 of Part I holds with

$$\begin{aligned} p &= r_{\min} + 1 \\ q &= r_{\min} + 1 + \alpha, \qquad \text{where } \alpha \in [0,1]. \end{aligned}$$

This implies that the average case complexity of $\rho\mathrm{INT}_d$ is bounded by

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon; \rho\mathrm{INT}_d)) \,=\, O\left(c\,\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1)} \left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(1+\alpha(\varepsilon))}\right),\ \alpha(\varepsilon) \in [0, \tfrac{1}{r_{\min}+1}]. \tag{5.1}$$

As explained in Corollary 4.3 of Part I, for some weights ρ, (5.1) is sharp. Observe that the exponent of $\rho\mathrm{INT}$ is at most 1.

The sample points and the linear U_ρ for which the upper bound of (5.1) is achieved are given by (4.10) of Part I. In our case, this means that

$$\begin{aligned} U_\rho(f) \,&=\, \sum_{j=1}^n f(x_j^*) \int_D \rho(t)\, h_j^*(t)\, dt, \\ n \,&=\, O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1)} \left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(1+1/(r_{\min}+1))}\right), \end{aligned} \tag{5.2}$$

where the sample points x_j^*, and the functions h_j^* are given by (3.8).

From (5.1) and (5.2) we conclude the following theorem.

Theorem 5.1. *Multivariate weighted integration ρINT with respect to the folded Wiener sheet measure is tractable and has exponent at most* 1.

Multivariate weighted integration $\rho\mathrm{INT}_d$ can be solved by using the sample points x_j^* and the linear U_ρ of (5.2). There exist weights ρ, so that (5.2) is optimal modulo a power of $\log 1/\varepsilon$.

We now show that the power of $\log 1/\varepsilon$ in (5.1) can be improved for multivariate integration, INT_d, i.e., for the weight $\rho(t) \equiv 1$ by using sample points which are different than the points x_j^* derived from hyperbolic cross points.

5.1. Multivariate Integration and Discrepancy

Assume that $\vec{r} = \vec{0}$. That is, $F = C^{\vec{0}}$ is now the class of continuous functions and $\mu = w$ is the classical Wiener measure. It turns out that, in this case, multivariate integration in the average case setting is related to *discrepancy*, see Woźniakowski [91]. The minimal average case error e_n given by (4.3) of Part I is, modulo a multiplicative factor which may depend on d, equal to the minimal discrepancy in the L_2 norm of n points. The latter is known due to Roth [54, 80], see also Frolov [80] for a sharp upper bound. This leads to $e_n = \Theta\big(n^{-1}\,(\log n)^{(d-1)/2}\big)$ and

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon; \mathrm{INT}_d) \;=\; \Theta\left(c\,\frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon}\right)^{(d-1)/2}\right). \tag{5.3}$$

Hence, the power of $\log 1/\varepsilon$ is decreased from at least $d-1$ in (5.1) to $(d-1)/2$ in (5.3). This shows that the weight $\rho(t) \equiv 1$ is not the hardest weight.

As mentioned in Woźniakowski [91], the algorithm that achieves the average case complexity bound is not fully constructive. On the other hand, if Hammersley points are taken as the sample points x_j then the linear approximation

$$U(f) \;=\; \frac{1}{n}\sum_{j=1}^{n} f(x_j)$$

has average error at most ε provided that $n = O\left(\varepsilon^{-1}(\log\varepsilon^{-1})^{d-1}\right)$. Thus, as long as we can compute the number of sample points, we have a fully constructive algorithm whose cost slightly exceeds the average case complexity. The latter formula for n cannot be used, however, due to the unspecified multiplicative factor in the big O notation.

We now show that the number n can be computed using the formula for the average error of U. Let

$$R_n(t) \;=\; R_n(t; z_1, \dots, z_n) \;=\; n^{-1} \sum_{k=1}^{n} \chi_{[0,t)}(z_k) \;-\; t_1 t_2 \cdots t_d,$$

where $\chi_{[0,t)}$ be the characteristic (indicator) function of $[0,t)$. Here $[0,t) = [0,t_1) \times \cdots \times [0,t_d)$ is the d dimensional rectangle whose volume is $t_1 t_2 \dots t_d$. The points $z_k = \vec{1} - x_k$ with $\vec{1} = [1,1,\dots,1]$. Then

$$e^{\text{avg}}(U)^2 \;=\; \int_D R_n^2(t)\,dt \;=\; 3^{-d} - \frac{2}{n}\sum_{k=1}^{n}\prod_{i=1}^{d} x_{k,i}(1 - x_{k,i}/2) + \frac{1}{n^2}\sum_{k,p=1}^{n}\prod_{i=1}^{d}\min(x_{k,i}, x_{p,i}),$$

where $x_k = [x_{k,1}, \dots, x_{k,d}]$. Thus, $e^{\text{avg}}(U)$ can be computed exactly using $\Theta(n^2)$ comparisons and arithmetic operations. If $e^{\text{avg}}(U)$ is not less than ε we can increase n, say by doubling it, and try again with the new n. This would lead finally to an n for which the average error is no greater than ε but the cost $\Theta(n^2)$ of computing $e^{\text{avg}}(U)$ may well exceed the cost $c\,n$ of computing n function values.

Note, however, that $e^{\text{avg}}(U)$ can be approximately computed by using the Monte Carlo algorithm. Since $e^{\text{avg}}(U)^2$ is the integral of R_n^2 we can compute

$$\text{MC}(\vec{u}) \;=\; \frac{1}{k}\sum_{j=1}^{k} R_n^2(u_j), \quad \vec{u} = [u_1, \dots, u_k],$$

with independently and uniformly distributed u_j over D. Observe that the cost of computing $R_n(u)$ is of order n, and $\text{MC}(\vec{u})$ can be computed at cost $\Theta(k\,n)$. We have

$$\int_{D^k} \left(e^{\text{avg}}(U)^2 - \text{MC}(\vec{u})\right)^2\,d\vec{u} \;\le\; \frac{1}{k}\int_D R_n^4(t)\,dt \;\le\; \frac{1}{k}\,D_n^4,$$

where $D_n = \sup_{t\in D}|R_n(t)|$ is the discrepancy in the sup norm. Halton [60] proved that for Hammersley points, $D_n = O\left(n^{-1}(\log n)^{d-1}\right)$. Since $e^{\text{avg}}(U) = \Omega\left(n^{-1}(\log n)^{(d-1)/2}\right)$ we have

$$\int_{D^k}\left(1 - \frac{\text{MC}(\vec{u})}{e^{\text{avg}}(U)^2}\right)^2 d\vec{u} \;=\; O\left(\frac{1}{k}\,(\log n)^{2(d-1)}\right).$$

Thus, to get $\text{MC}(\vec{u})$ of order $e^{\text{avg}}(U)^2$ we should take $k = O\left((\log n)^{2(d-1)}\right)$. Then the total cost of computing $\text{MC}(\vec{u})$ is $O\left(n(\log n)^{2(d-1)}\right)$. For large n, or equivalently for small ε, this is smaller than $\Theta(n^2)$.

5.2. Multivariate Integration for Folded Wiener Sheet Measures

In this section we briefly discuss multivariate integration for the general case of folded Wiener sheet measures.

Multivariate integration for the unit ball $BW_2^{\overline{r+1}}$ in the worst case setting has been analyzed by Bykovskij [85] and Frolov [76]. Frolov obtained upper bounds for $r_i \equiv r_{\min}$, whereas Bykovskij studied arbitrary r_i and obtained lower and upper bounds. These results yield that e_n of (4.3) of Part I is now of the form

$$e_n = \Theta\left(\frac{1}{n^{r_{\min}+1}} (\log n)^{(k^*-1)/2}\right).$$

The average case complexity, see Paskov [91], is given by

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon; \mathrm{INT}_d) = \Theta\left(c \left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1)} \left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)/(2r_{\min}+2)}\right). \tag{5.4}$$

Hence, also in this case, the power of $\log 1/\varepsilon$ is decreased from at least (k^*-1) in Corollary 5.1 to $(k^*-1)/(2r_{\min}+2)$ in (5.4).

Sample points and an algorithm whose cost is equal to the average case complexity modulo a multiplicative factor which may depend on d or modulo a power of $\log 1/\varepsilon$ are given in Paskov [91]. (These sample points are different from the points x_j^* of (3.8).)

6. Multivariate Function Approximation

In this section we apply the results of Part I and the worst case estimates of Section 3 for multivariate function approximation $\mathrm{APP} = \{\mathrm{APP}_d\}$ in the average case setting,

$$\mathrm{APP}_d = \{F, \mu, L_2(D), I_d, \Lambda^{\mathrm{std}}\}.$$

where F and μ are given by (2.1) and (2.2). Obviously, (A.1) and (A.2) hold with $K_1 = 1$ and $K_2 \le 1$.

As explained in Section 4.2 of Part I, we can use two approaches to study APP_d. For the first approach, we can check that (2.7) and (3.10) imply that (4.13) of Part I holds. This yields, modulo a power of $\log 1/\varepsilon$, the average case complexity of APP_d.

The second approach is slightly better since it allows us to determine the average case complexity of APP_d with the proper power of $\log 1/\varepsilon$. Therefore we only discuss the second approach. In this approach we use the worst case results on multivariate function approximation $\mathrm{APP}_d^{\mathrm{wor},\infty}$. From (3.9) we conclude that Theorem 4.3 of Part I holds with

$$p_2 \;=\; 1/(r_{\min}+1/2) \quad \text{and} \quad q_2 \;=\; (k^*-1)(r_{\min}+1).$$

Thus, Theorem 4.3 of Part I states that

$$\mathrm{comp}^{\mathrm{avg}}(\varepsilon;\mathrm{APP}_d) \;=\; O\left(c\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1/2)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(r_{\min}+1)/(r_{\min}+1/2)}\right). \tag{6.1}$$

Since $1/p_1 - 1/p_2 = 1/2$, the lower bound from Theorem 4.3 of Part I does not match (6.1). As mentioned before, we can bound $\mathrm{comp}^{\mathrm{avg}}(\varepsilon;\mathrm{APP}_d)$ by the corresponding average case complexity in Λ^{all}, which is known due to Papageorgiou and Wasilkowski [90], see (4.1). These two average case complexity functions may differ by only a multiplicative factor depending on d. This shows that the lower bound of Theorem 4.3 of Part I is not sharp.

More importantly, $p_2 = p$ and Corollary 4.5 of Part I yields that the big O in (6.1) can be replaced by Θ. Furthermore, the linear approximation T_n^* given by (3.8) is almost optimal, i.e., its average error is minimal modulo a multiplicative factor which may depend on d. The average cost T_n^* is $(c+2)n$. Hence, it nearly minimizes the average cost of computing an ε-approximation if

$$n \;=\; O\left(\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1/2)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(r_{\min}+1)/(r_{\min}+1/2)}\right). \tag{6.2}$$

We summarize this in the following theorem.

Theorem 6.1. *For multivariate function approximation APP_d with respect to the folded Wiener sheet measure, the average case complexity functions $comp^{\mathrm{avg}}(\varepsilon;\Lambda^{\mathrm{std}})$ and $comp^{\mathrm{avg}}(\varepsilon;\Lambda^{\mathrm{all}})$ differ at most by a multiplicative factor which may depend on d and*

$$comp^{\mathrm{avg}}(\varepsilon;\Lambda^{\mathrm{std}}) \;=\; \Theta\left(c\left(\frac{1}{\varepsilon}\right)^{1/(r_{\min}+1/2)}\left(\log\frac{1}{\varepsilon}\right)^{(k^*-1)(r_{\min}+1)/(r_{\min}+1/2)}\right).$$

The average cost of the linear T_n^* given by (3.8) which uses n sample points derived from the hyperbolic cross points with n given by (6.2) is equal to the average case complexity functions in Λ^{std} and Λ^{all} modulo a multiplicative factor which may depend on d.

We stress that the computation of $T_n^*(f)$ does *not* require any precomputation and can be readily implemented. We plan to test T_n^* in the near future.

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