Finiteness of the Set of Frames of Graphs of Bounded Average Genus

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Abstract

In this report, a detailed proof is given that for every real number r, there is a finite set $\mathcal{F}_{<r}$ of graphs, which are called "frames", such that every cutedge-free smooth graph whose average genus is less than r can be obtained by attaching a set of non-overlapping ears to some member of $\mathcal{F}_{<r}$.

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1 Preliminaries

In this report, we give a detailed proof that for every real number r, there is a finite set $\mathcal{F}_{<r}$ of graphs called "frames" such that every cutedge-free smooth graph whose average genus is less than r can be obtained by attaching a "set of non-overlapping ears" to some member of $\mathcal{F}_{<r}$. This result is used by Chen [1] in his derivation of a linear-time isomorphism test for graphs of bounded average genus and Chen and Gross [4] in their proof that the set of values of average genus has no lower limit points.

We assume that the reader is thoroughly familiar with topological graph theory, as described by Gross and Tucker [5] or by White [7].

A *graph* may have multiple adjacencies or self-adjacencies. An *imbedding* must have the "cellularity property" that the interior of every region is simply connected. The closed orientable surface of genus n is denoted S_n.

A *rotation* at a vertex v is a cyclic permutation of the edge-ends incident on v. Thus, a d-valent vertex admits $(d-1)!$ rotations. A list of rotations, one for each vertex of the graph, is called a *rotation system*.

An imbedding of a graph G in an orientable surface induces a rotation system, as follows: the rotation at vertex v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in a orientation-preserving tour around v. Conversely, by the Heffter-Edmonds principle, every rotation system induces a unique imbedding of G into an orientable surface. The bijectivity of this correspondence implies that the number of different ways to imbed a graph of valence sequence $d_1, \cdots, d_n$ into a closed, orientable surface is

$$\prod_{i=1}^{n}(d_i - 1)!$$

For any graph G, if the number of imbeddings in the surface S_k is denoted g_k, then the sequence

$$g_0, g_1, g_2, \cdots$$

is called the *genus distribution* of G. The *average genus* of G is defined to be the value

$$\gamma_{avg}(G) = \frac{\sum_{i=0} i \cdot g_i}{\sum_{i=0} g_i}$$

A graph is said to be *smooth* if it does not contain any 2-valent vertices. We observe that neither the operation of subdividing an edge nor the operation of "smoothing out" a 2-valent vertex changes the average genus. We

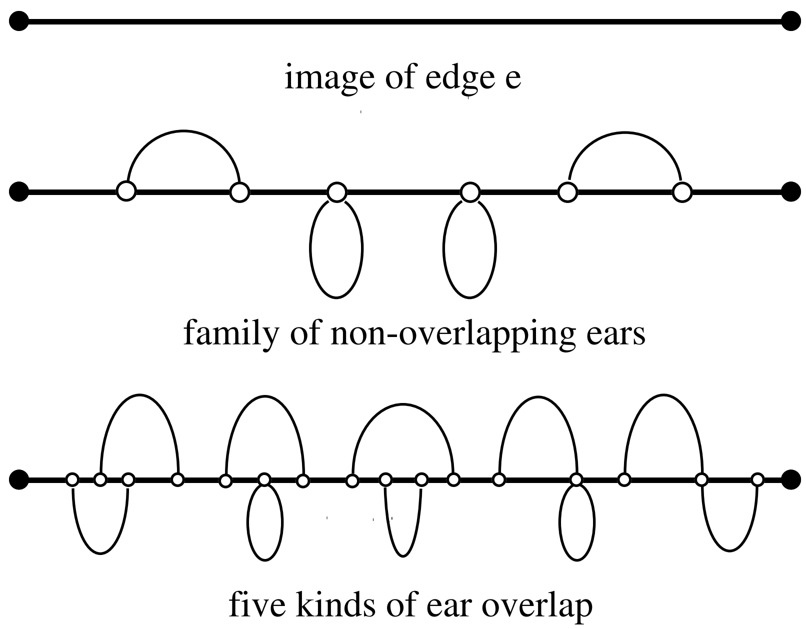

Figure 1: Overlapping and non-overlapping ear families.

also observe that every connected graph homeomorphism type but one has exactly one smooth representative. The cycle has no smooth representatives.

We say that we add an *open ear* to (the interior of) an edge e of a graph G if we insert two vertices in the interior of e and then double the edge between them. Similarly, we say that we add a *closed ear* if we insert one new vertex in the interior of edge e and then attach a self-loop at that new vertex.

Let $h: H \to G$ be a graph homeomorphism, and let e be an edge of H, so that the image $h(e)$ of the edge e is a path in G. A subset $\mathcal{U}$ of the bridges for H in G is said to be a *family of non-overlapping ears* on the image of the edge e under the following conditions (as illustrated in Figure 1):

1. Each bridge in $\mathcal{U}$ is an ear on the image of e.
2. No two ears in $\mathcal{U}$ share a point of attachment to the image of e.
3. No ear in $\mathcal{U}$ has an attachment between the attachments of another ear in $\mathcal{U}$.

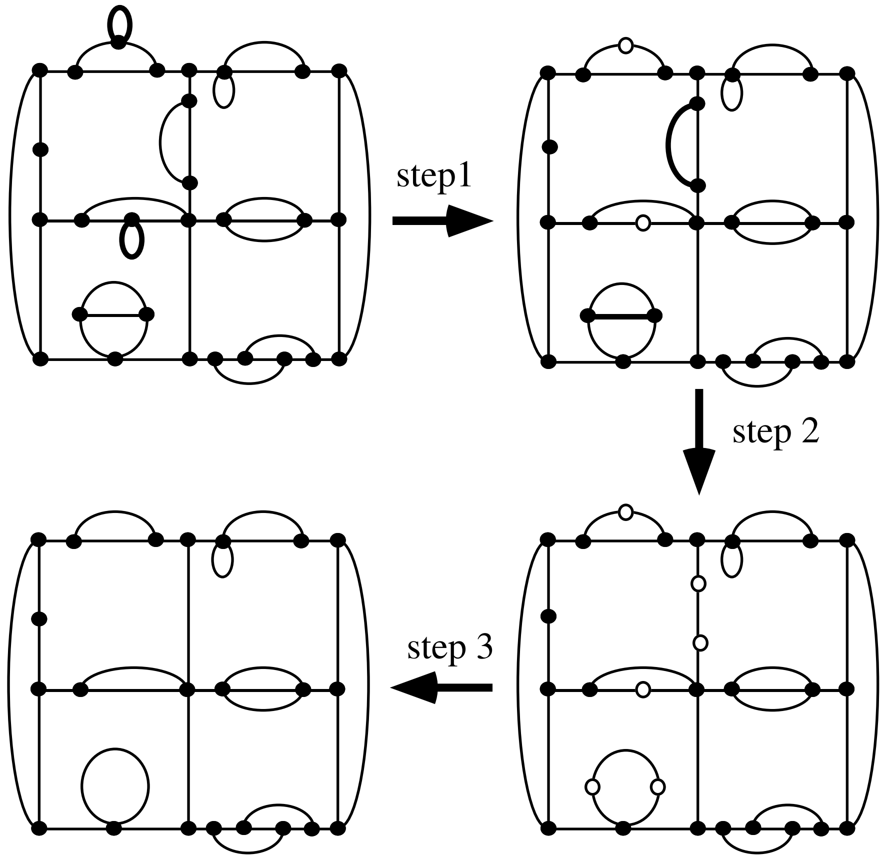

Figure 2: Constructing the frame of a graph G.

For the sake of intelligible discussion, various minor variations of terminology are introduced from time to time in the discussion of bridge and ear constructions, without explicitly belaboring the meaning.

Let G be a graph. The *frame* of G, denoted *Fr(G)*, is the graph obtained by first deleting a maximal set of non-overlapping ears from G and then smoothing the resulting 2-valent vertices. More precisely, it is the (not necessarily smooth) graph obtained by the following sequence of steps (see Figure 2 for an illustration):

1. Delete every self-loop in G whose endpoint is of valence exactly 4.

2. For each pair of 3-valent vertices of G that are doubly adjacent to each

other, delete one of the two edges that join them.

3. Smooth out all 2-valent vertices in the graph that results from Steps 1 and 2 (but not other 2-valent vertices).

Note that every graph H is a frame of some graph. A graph G such that $Fr(G) = H$ can be simply obtained by adding a closed ear to every edge of H. Other two easily verified facts are that the frame of a cutedge-free graph is also cutedge-free and that the frame of a smooth graph is also smooth.

The following theorem is frequently applied herein, and its proof can be found in [2].

Theorem 1.1 *The average genus of a graph is not less than the average genus of any of its subgraphs.*

2 The main theorem

Notation. $\mathcal{F}_{<r}$ denotes the set of graphs that are frames of cutedge-free smooth graphs of average genus less than r.

The objective of this section is to prove that the set $\mathcal{F}_{<r}$ is finite, for every real number $r \geq 0$.

We use Tutte's theory of bridges [6], using the slightly modified terminology introduced in [3]. The following terms are defined in §4 of [3]: *vertex of attachment, trivial bridge, non-trivial bridge, smooth model.*

Suppose that G is a cutedge-free smooth graph and that H is a cutedge-free subgraph of G. We begin with three lemmas that relate the numerical difference $\gamma_{avg}(G) - \gamma_{avg}(H)$ to the graphical difference $G - H$. Similar lemmas for the category of 2-connected graphs were proven in a preceding paper [3], and the extension of those proofs to the category of cutedge-free graphs is straightforward. Therefore, we only sketch the proofs here and refer the interested reader to [3] for supplementary details.

Lemma 2.1 *Let G be a cutedge-free smooth graph and H a cutedge-free subgraph of G. Suppose that B is a non-trivial bridge of $G - H$. Then there are two simple paths P and P' in B such that P and P' have no common interior vertices, that $H \cup \{P, P'\}$ is cutedge-free, and that*

$$\gamma_{avg}(H \cup \{P, P'\}) - \gamma_{avg}(H) \geq \frac{1}{3}$$

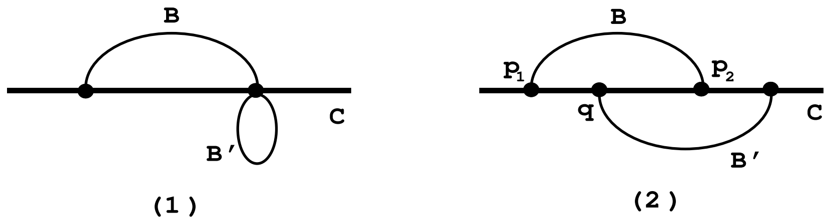

Figure 3: Two trivial bridges B and B' overlapping on a chain C

PROOF. Since G is cutedge-free and smooth, we can find a simple path P in B whose end-vertices, which may be identical, are attachments of H. The path P cannot be a single edge, since the bridge B is non-trivial. Moreover, no interior vertex of P can be 2-valent in G, since G is smooth. Therefore, we must be able to find another simple path P' in B from an interior vertex of the path P to another vertex, which is either an interior vertex of P or an attachment of the subgraph H, such that P and P' have no common interior vertices. Note that the two end-vertices of P' may also be identical.

The graph $H \cup \{P, P'\}$ is obviously cutedge-free. Moreover, as shown in the proof of Lemma 4.1 of [3], we have

$$\gamma_{avg}(H \cup \{P, P'\}) - \gamma_{avg}(H) \geq \frac{1}{3} \qquad \square$$

A *chain* C of a graph H (not necessarily smooth) is a simple path in H such that every interior vertex of C has valence 2 in H. A chain is *maximal* if its two end-vertices have valence greater than 2 in H.

Let H be a cutedge-free subgraph of a cutedge-free graph G. Two trivial bridges in $G - H$ *overlap on a chain C of H* if either of the following two cases holds: 1) They share a common attachment that is an interior vertex of C; 2) One of them has two different endpoints p_1 and p_2 on the chain C, and the other has an endpoint q between p_1 and p_2 on the chain C such that q is an interior vertex of C. Figure 3 partially illustrates the two different cases for two trivial bridges to overlap. In Case 2, the other end of bridge B' need not lie on chain C. In Case 1, the other ends of bridges B and B' need not lie on chain C.

Lemma 2.2 *Let G be a cutedge-free smooth graph and H a cutedge-free subgraph of G. Suppose that there are two trivial bridges B and B' in $G - H$*

overlapping on a chain of H. Then the graph $H \cup \{B, B'\}$ is cutedge-free and

$$\gamma_{avg}(H \cup \{B, B'\}) - \gamma_{avg}(H) \geq \frac{1}{3}$$

PROOF. The graph $H \cup \{B, B'\}$ is obviously cutedge-free.

In Case 1, if the two trivial bridges B and B' share a common attachment that is an interior vertex of a chain C in H, then as shown in the proof of Lemma 4.2 of [3], we have

$$\gamma_{avg}(H \cup \{B, B'\}) - \gamma_{avg}(H) \geq \frac{1}{3}$$

In Case 2, suppose that the bridge B has two different endpoints p_1 and p_2 on the chain C, and B' has an endpoint q between p_1 and p_2 on C such that $q \neq p_1, p_2$. Consider the graph $H' = H - C' + B$, which is obtained by deleting the subchain C' adjoining p_1 and p_2 on C, then adding the edge B to the resulting graph. Since C' and B' are two simple paths in $G - H'$ such that the two endpoints of C' are attachments of H' and one of the endpoints of B' is at an interior vertex of C', by the proof of Lemma 4.1 of [3], we have

$$\gamma_{avg}(H' \cup \{C', B'\}) - \gamma_{avg}(H') \geq \frac{1}{3}$$

Furthermore, it is obvious to see that the graph H' is homeomorphic to the graph H, and that $H \cup \{B, B'\} = H' \cup \{C', B'\}$. Therefore,

$$\gamma_{avg}(H \cup \{B, B'\} - \gamma_{avg}(H) = \gamma_{avg}(H \cup \{C', B'\} - \gamma_{avg}(H') \geq \frac{1}{3} \qquad \square$$

Let H be a cutedge-free subgraph of a cutedge-free graph G. Suppose that C and C' are two distinct maximal chains in H. We say that a trivial bridge B in $G - H$ *links the interiors of C and C'* if one attachment of B is an interior vertex of C and the other attachment of B is an interior vertex of C'.

Let C and C' be chains in H. Let B_1, B_2, $\cdots$, B_h be h trivial bridges of $G - H$, such that each bridge B_i has an attachment a_i and b_i on C and C', respectively, for $i = 1, 2, \cdots, h$. We say that the bridges B_1, B_2, $\cdots$, B_h link the interiors of the two chains C and C' *in order* if we can travel the attachments on the chains C and C' in the respective orderings a_1, a_2, $\cdots$, a_h, and b_1, b_2, $\cdots$, b_h. We also require that these bridge attachments are distinct. We observe that there are two ways to link the interiors of chains C and C' in order, depending on the direction in which C and C' are traversed.

If three bridges link the interiors of C and C' but do not link them in order, we say they link them *askew*. For instance, if any two bridges share an attachment, then they are askew.

Lemma 2.3 *Let G be a cutedge-free smooth graph and H a cutedge-free subgraph of G. Suppose that there are three trivial bridges B, B', and B'' in $G - H$ linking the interiors of two chains C and C' in H askew. Then the graph $H \cup \{B, B', B''\}$ is cutedge-free and*

$$\gamma_{avg}(H \cup \{B, B', B''\}) - \gamma_{avg}(H) \geq \frac{1}{4}$$

PROOF. Again it is obvious that the graph $H \cup \{B, B', B''\}$ is cutedge-free.

If any two of the bridges B, B', and B'', say B and B', share a common attachment on a maximal chain C of H, then B and B' have a Case 1 overlap on C. Thus, by Lemma 2.2, we have $\gamma_{avg}(H \cup \{B, B'\}) - \gamma_{avg}(H) \geq \frac{1}{3}$. Therefore, by Theorem 1.1

$$\gamma_{avg}(H \cup \{B, B', B''\}) - \gamma_{avg}(H) \geq \gamma_{avg}(H \cup \{B, B'\}) - \gamma_{avg}(H) \geq \frac{1}{3} > \frac{1}{4}$$

Thus, without loss of generality, we can assume that all endpoints of B, B', and B'' are distinct. As shown in the proof of Lemma 4.4 of [3], if three bridges B, B', and B'' in $G - H$ link the interiors of two maximal chains in H askew, then

$$\gamma_{avg}(H \cup \{B, B', B''\}) - \gamma_{avg}(H) \geq \frac{1}{4} \qquad \square$$

With these preparations, we are ready for the following crucial lemma, which we need in the proof of finiteness of the set of frames of bounded average genus.

Lemma 2.4 *Let G be a cutedge-free smooth graph of average genus less than r, $r > 0$. Then G has a cutedge-free subgraph H such that all bridges of $G - H$ are trivial, that no two trivial bridges of $G - H$ overlap on a chain of H, and that no three trivial bridges in $G - H$ link the interiors of two* *chains in H askew. Moreover, the smooth model of H contains fewer than $24r$ vertices and fewer than $36r$ edges.*

PROOF. We start by selecting a subgraph H_0 of G that is a simple cycle. Then we construct an increasing sequence of cutedge-free subgraphs H_i, $i = 0, 1, 2, \cdots$, of G inductively. In particular, suppose that we have obtained a subgraph H_i of G. Then we augment H_i in any one of the following three ways, if any of them is applicable, to produce the subgraph H_{i+1}.

1. If there is a non-trivial bridge B in $G - H_i$, then we find two paths P_i and P_i' in B as in Lemma 2.1, and we let $H_{i+1} = H_i \cup \{P_i, P_i'\}$.
2. If there are two trivial bridges B_i and B_i' in $G - H_i$ that overlap on a chain of H_i, then we let $H_{i+1} = H_i \cup \{B_i, B_i'\}$.
3. If there are three trivial bridges B_i, B_i', and B_i'' in $G - H_i$ that link the interiors of two chains in H_i askew, then we let $H_{i+1} = H_i \cup \{B_i, B_i', B_i''\}$.

Since each H_i constructed above is a subgraph of the graph G, it follows from Theorem 1.1 that the average genus of H_i is less than r. Moreover, by Lemma 2.1, Lemma 2.2, and Lemma 2.3, the increment in average genus from H_i to H_{i+1} is at least $1/4$. Therefore, there is an index I, $I < 4r$, such that none of the above conditions apply to the graph H_I. Moreover, since the graph H_I is obtained by inserting at most $3I$ paths starting from the simple cycle H_0, it is easy to prove by induction that the smooth model of H_I contains at most $2 \cdot 3I < 24r$ vertices and at most $3 \cdot 3I < 36r$ edges. $\square$

Recall that a graph with two vertices and n edges adjoining them is called a *dipole* and is denoted D_n. Also, a graph with one vertex and n self-loops is called a *bouquet* and is denoted B_n. Finally, an *n-rung closed-end ladder* L_n is obtained by taking the graphical Cartesian product of an n-vertex path with the complete graph K_2, and then doubling both its end edges. The proof for the following theorem can be found in [2] (Lemma 5.5 and Lemma 5.6) and [3] (Corollary 2.3).

Theorem 2.5 *For every sufficiently large number n,*

$$\gamma_{avg}(D_n) \geq \frac{n}{4} \qquad \gamma_{avg}(B_n) \geq \frac{n}{4} \qquad \gamma_{avg}(L_n) \geq \frac{n}{3}$$

Now we are ready for the main theorem of this section. Recall that we denote by $\mathcal{F}_{<r}$ the set of frames of cutedge-free smooth graphs of average genus less than r.

Theorem 2.6 *For every real number $r \geq 0$, the set $\mathcal{F}_{<r}$ is finite.*

PROOF. It is easy to see that if the theorem holds for some real number $r > 0$, then the theorem also holds for all real numbers r' with $0 \leq r' \leq r$, since by its definition the set $\mathcal{F}_{<r'}$ is a subset of the set $\mathcal{F}_{<r}$. Therefore, without loss of generality, we can assume that the number r is a positive integer such that for every integer $n \geq r$, Theorem 2.5 holds. It is now sufficient to prove that there is a constant c_r such that every frame in $\mathcal{F}_{<r}$ has at most c_r edges.

Let F be an arbitrary frame in the set $\mathcal{F}_{<r}$, which is obtained by deleting a maximal set of non-overlapping ears from a cutedge-free smooth graph G of average genus less than r.

By Lemma 2.4, we can find a cutedge-free subgraph H of G such that all bridges in $G - H$ are trivial bridges, no two trivial bridges in $G - H$ overlap on a chain of H, and no three trivial bridges in $G - H$ link the interiors of two chains in H askew. Moreover, the smooth model H_s of H contains fewer than $24r$ vertices and fewer than $36r$ edges. We observe that the number of vertices and the number of edges of H_s equal the number of vertices of valence > 2 and the number of different maximal chains of H, respectively.

Every bridge e of $G - H$ must lie in one of the following six possible classes (a trivial bridge is called an *open bridge* if it is not a self-loop):

1. The bridge e is a self-loop and is attached to a vertex of valence greater than 2 in H.

2. The bridge e is an open bridge with its two endpoints attached to two vertices both of valence greater than 2 in H.

3. The bridge e is an open bridge and links the interiors of two different chains in H.

4. The bridge e is an open bridge such that one of its endpoint is attached to a vertex v of valence greater than 2 in H, and the other endpoint of it is attached to an interior vertex of a maximal chain C in H.

5. The bridge e is an open bridge such that its two endpoints are attached to two adjacent 2-valent vertices of H.

6. The bridge e is a self-loop and is attached to an interior vertex of a chain in H.

We consider how many edges of $G - H$ can be in Classes 1-4.

Suppose that $G - H$ contains more than $96r^2$ trivial bridges in Class 1. Since the number of vertices of valence greater than 2 in H equals the number of vertices in its smooth model H_s, which is less than $24r$, there must be a vertex v in H such that in G there are more than $4r$ self-loops located at v. Consequently, G contains a bouquet B_h of h circles with $h > 4r$. By Theorem 2.5, B_h has average genus larger than r. This contradicts Theorem 1.1. Therefore, there are at most $96r^2$ edges of $G - H$ belonging the Class 1.

Similarly, no pair of vertices of valence > 2 in H can be adjoined by more than $4r$ edges in G, otherwise G would contain a dipole D_h with $h > 4r$, so we would have $\gamma_{avg}(G) \geq \gamma_{avg}(D_h) > r > \gamma_{avg}(G)$, by Theorem 1.1, Theorem 2.5 and our assumption. Moreover, H has less than $\binom{24r}{2}$ distinct pairs of vertices of valence > 2. Thus, the number of edges in Class 2 is at most $4r\binom{24r}{2}$.

For a pair (C, C') of two distinct maximal chains of H, suppose that there are $h > 3r + 2$ edges in $G - H$ linking the interiors of C and C'. Since no two edges in $G - H$ overlap on a chain of H, all endpoints of these h edges are distinct. Moreover, since no three edges in $G - H$ link the interiors of C and C' askew, these h edges link the interiors of C and C' in order. Consequently, a subgraph of G, which consists of $3r + 3$ of these h edges and the chains C and C', is homeomorphic to the $(3r + 1)$-rung closed-end ladder L_{3r+1}, which has average genus greater than r by Theorem 2.5. This again contradicts Theorem 1.1. Since there are at most $\binom{36r}{2}$ distinct pairs of maximal chains in H, we conclude that the number of edges in Class 3 is at most $(3r + 2)\binom{36r}{2}$.

Finally, suppose that there are more than $8r$ edges of $G - H$ that have one endpoint on a vertex v of valence > 2 in H and the other endpoint on an interior vertex of a maximal chain C of H. Since no two of these edges have a Case 1 overlap on C, these edges together with the vertex v and the maximal chain C would contain a subgraph that is homeomorphic to B_h, where $h \geq 4r$. Thus, by Theorem 2.5, the graph G would contain a subgraph of average genus at least $h/4 \geq r > \gamma_{avg}(G)$, contradicting Theorem 1.1. This proves that $G - H$ contains at most $8r \cdot 24r \cdot 36r$ edges in Class 4.

Adding to H all the trivial bridges in $G - H$ that are in Classes 1-4, we obtain a cutedge-free graph H'. Let H'_s be the smooth model of H'. We calculate the number of edges in H'_s by restoring edges to H_s. Restoring an edge in Class 1 or in Class 2 increases the number of edges by exactly 1. Restoring an edge in Class 3 increases the number of edges by exactly 3, and restoring an edge in Class 4 increases the number of edges by exactly 2. By the analysis above, the graph H'_s is reconstituted by adding at most $96r^2$ edges in Class 1, at most $4r\binom{24r}{2}$ edges in Class 2, at most $(3r+2)\binom{36r}{2}$ edges in Class 3, and at most $8r \cdot 24r \cdot 36r$ edges in Class 4, to the graph H_s, which has at most $36r$ edges. We conclude that the number of edges in H'_s is at most

$$c_r = 36r + 96r^2 + 4r\binom{24r}{2} + 3 \cdot (3r+2)\binom{36r}{2} + 2 \cdot 8r \cdot 24r \cdot 36r$$

Now each trivial bridge in $G - H'$ is either a self-loop at a 2-valent vertex of H', or an open bridge such that its two endpoints are attached to two adjacent 2-valent vertices in H'. Moreover, no two trivial bridges in $G - H'$ share an attachment. Consequently, the graph G can be obtained by attaching non-overlapping ears to the smooth model H'_s of the graph H'. In other words, the graph H'_s can be obtained by deleting non-overlapping ears from G.

Since the frame $F \in \mathcal{F}_{<r}$ is obtained by deleting a maximal set of non-overlapping ears from G, the frame F is homeomorphic to a subgraph of the graph H'_s. Now both F and H'_s are smooth graphs, so the number of edges of the graph F cannot be larger than the number of edges of the graph H'_s, which is at most c_r.

Therefore, we have proved that every graph in the set $\mathcal{F}_{<r}$ has at most c_r edges. It follows that the set $\mathcal{F}_{<r}$ is finite. □

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