# Limit Points for Average Genus (II) : 2-Connected Non-simplicial Graphs

Jianer Chen[1]

Computer Science Department

Texas A&M University

College Station, TX 77843-3112

Jonathan L. Gross[2]

Department of Computer Science

Columbia University

New York, NY 10027

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Abstract

*In part (I) of this paper, it is proved that there are no limit points for the set of values of average genus of 2-connected simplicial graphs and of 3-connected graphs. The need for such restrictions is now demonstrated by showing that infinitely many real numbers are limit points of values of average genus for 2-connected non-simplicial graphs. A systematic method for constructing such limit points is presented, and it is proved that this method is essentially the only way to construct limit points of values of average genus for "homeomorphically nested" 2-connected graphs.*

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# 1 Introduction

In part (I) of this paper [ChGr 1990c], we have examined the values of average genus for 2-connected simplicial graphs and for 3-connected graphs. We proved that in each finite real interval, there are only finitely many real numbers that are values of average genus of 2-connected simplicial graphs or of 3-connected graphs; moreover, for each fixed real number $r$, there are only finitely many 2-connected simplicial graphs or 3-connected graphs sharing $r$ as the value of their average genus. Consequently, there are no limit points for values of average genus of 2-connected simplicial graphs and of 3-connected graphs.

Obviously, these results cannot be true for arbitrary connected graphs. For example, there are infinitely many non-homeomorphic trees, all with average genus zero. Moreover, there are infinitely many 2-connected graphs of average genus less than 1 and infinitely many different possible values of average genus less than 1. Indeed, the number 1 is actually an upper limit point of values of average genus of 2-connected graphs, as demonstrated by Gross, Klein, and Rieper [GrKlRi 1989]. Therefore, some natural questions to ask are whether the number 1 is just an exceptional point; or if not, what kind of real number can be a limit point of values of average genus of 2-connected graphs; how many such limit points are there; and how are these limit points constructed?

The main purpose of this second installment is to investigate the limit points of values of average genus of 2-connected graphs. We first give a systematic method, which is a generalization of Klein's construction for necklaces, to construct limit points of values of average genus of 2-connected graphs. We prove that there are infinitely many limit points of values of average genus of 2-connected graphs. In fact, in each real interval of length 1 on the real line, there are at least two such limit points. Then we explore the properties of the limit points of values of average genus of 2-connected graphs and show that our necklace-like construction is essentially the only way to construct limit points of values of average genus of 2-connected graphs.

Some of our proofs are based on the beautiful theory of bridges and attachments, as presented by Tutte [Tutte 1966] in his classic study of graph connectivity.

Now we briefly review the fundamentals of topological graph theory. It is assumed that the reader is somewhat familiar with it. For further description, see Gross and Tucker [GrTu 1987] or White [White 1984].

A *graph* may have multiple adjacencies or self-adjacencies. An *imbedding* must have the "cellularity property" that the interior of every region is simply connected. The closed orientable surface of genus $n$ is denoted $S_n$.

A *rotation* at a vertex $v$ is a cyclic permutation of the edge-ends incident on $v$. Thus, a $d$-valent vertex admits $(d-1)!$ rotations. A list of rotations, one for each vertex of the graph, is called a *rotation system.*

An imbedding of a graph $G$ in an orientable surface induces a rotation system, as follows: the rotation at vertex $v$ is the cyclic permutation corresponding to the order in which the edge-ends are traversed in a orientation-preserving tour around $v$. Conversely, by the Heffter-Edmonds principle, every rotation system induces a unique imbedding of $G$ into an orientable surface. The bijectivity of this correspondence implies that the number of different ways to imbed a graph of valence sequence $d_1, \cdots, d_n$ into a closed, orientable surface is

$$\prod_{i=1}^{n}(d_i-1)!$$

For any graph $G$, if the number of imbeddings in the surface $S_k$ is denoted $g_k$, then the sequence

$$g_0, g_1, g_2, \cdots$$

is called the *genus distribution* of $G$. Genus distributions for special families of graphs have been calculated by Furst, Gross, and Statman [FuGrSt 1989] and by Gross, Robbins, and Tucker [GrRoTu 1989], by blending topology and combinatorial enumeration.

The *average genus* of $G$ is defined to be the value

$$\gamma_{avg}(G) = \frac{\sum_{i=0} i \cdot g_i}{\sum_{i=0} g_i}$$

Average genus of individual graphs is in the Gross-Furst hierarchy [GrFu 1987] and studied by Gross, Klein and Rieper [GrKlRi 1989]. (Stahl [Stahl 1983] studied the related problem of average genus of a class of graphs with the same number of edges.)

The proof for the following theorem can be found in Chen and Gross [ChGr 1990c].

**Theorem 1.1** *The average genus of a graph is not less than the average genus of any of its subgraphs.*

A *necklace of type* $(d, 0)$ is a graph $N_{(d,0)} = (V, E)$, such that $V = \{v_1, v_2, \cdots, v_{2d}\}$, the vertex $v_{2j-1}$ is connected by a single edge to the vertex $v_{2j}$, $j = 1, \cdots, d$, and the vertex $v_{2j-2}$ is connected by two multiple edges to the vertex $v_{2j-1}$, where we let $v_0$ be $v_{2d}$. A *necklace of type* $(d, s)$ is the necklace $N_{(d,0)}$ of type $(d, 0)$ plus $s$ self-loops added to $s$ distinct interior points of those non-multiple edges of $N_{(d,0)}$ (in an arbitrary way). These necklaces were developed by Klein (see [GrKlRi 1989]).

The graphs in Figure 1 are two necklaces of type $(4, 3)$. Although there is a unique necklace of type $(d, 0)$ up to homeomorphism, there are more than one non-homeomorphic necklaces of type $(d, s)$ for $d > 1$ and $s > 1$.

The proof of the following theorem can be found in Gross, Klein and Rieper [GrKlRi 1989] (see also Chen and Gross [ChGr 1990b]).

**Theorem 1.2** *The average genus of a necklace $N_{(d,s)}$ of type $(d, s)$ equals $1 - \frac{2^{s-d}}{3^s}$.*

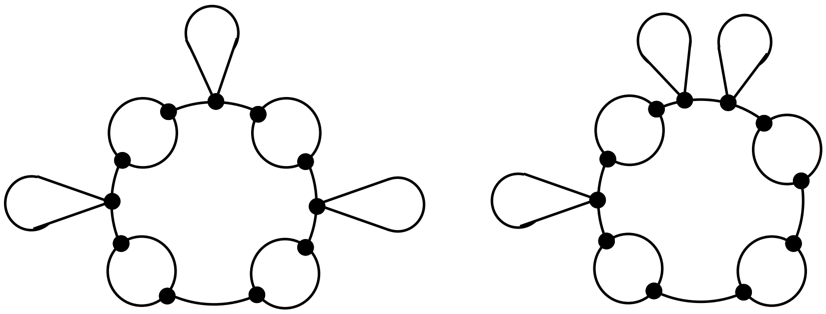

Figure 1: Two necklaces of type $(4, 3)$

The outline of this paper is as follows. Section 2 presents an example to show how to construct limit points of values of average genus of 2-connected graphs. Several interesting results are derived from this example. In Section 3, we systematically formulate the construction. Section 4 contains proof that our method is essentially the only way to construct limit points of values of average genus for 2-connected graphs. In Section 5, some open problems are proposed and discussed.

## 2 The average genus of eared ladders

In this section, we give an example to show how we can construct a limit point for the values of average genus for 2-connected graphs.

A *n-rung closed-end ladder* $L_n$ can be obtained by taking the graphical Cartesian product of a $n$-vertex path $P_n$ with the complete graph $K_2$, and then doubling both its end edges (call all these edges at ends of the ladder *end-rungs*). Figure 2 illustrates a 4-rung closed-end ladder.

Furst, Gross, and Statman have derived the explicit formula for the genus distribution of closed-end ladders [FuGrSt 1989]. Schwenk and White

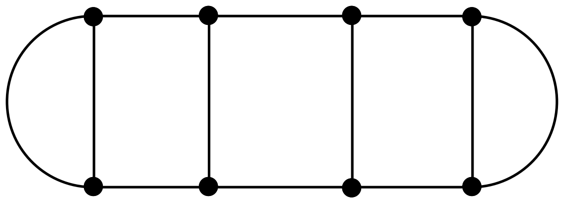

Figure 2: A 4-rung closed-end ladder

[ScWh 1990] (see also Lee and White [LeWh 1990]) computed the average genus of closed-end ladders from that formula. The present proof computes the average genus of closed-end ladders directly, without recourse to the formula. Indeed, our technique here, with a minor modification, gives us a new method, different from that of Furst, Gross, and Statman [FuGrSt 1989], to derive the genus distributions of closed-end ladders.

**Lemma 2.1** *Let $L_n$ be the closed-end ladder of $n$ rungs and let $e$ be an end-rung of $L_n$. Suppose that there are $d_n$ imbeddings of $L_n$ in which $e$ is a boundary of two different faces, and $s_n$ imbeddings of $L_n$ in which the two sides of $e$ lie in the same face. Then*

$$d_n = (-1)^n \frac{2^n}{3} + 2 \cdot \frac{4^n}{3} \qquad \text{and} \qquad s_n = (-1)^{n-1} \frac{2^n}{3} + \frac{4^n}{3}$$

*for $n \geq 1$.*

PROOF. The closed-end ladder $L_n$ can be obtained by attaching a new edge $\overline{e}$ to an end-rung $e$ of the closed-end ladder $L_{n-1}$ so that the two end-points of $\overline{e}$ are two distinct interior points of $e$. The edge $\overline{e}$ becomes an end-rung of the resulting closed-end ladder $L_n$. For each imbedding of $L_{n-1}$, there are four different ways to attach $\overline{e}$ to get an imbedding of $L_n$.

If the two sides of $e$ lie in the same face of an imbedding of $L_{n-1}$, then any way of attaching $\overline{e}$ to $e$ divides the face into two faces so $\overline{e}$ is a boundary of the two new faces. Therefore, corresponding to an imbedding of $L_{n-1}$ such that the two sides of $e$ lie in the same face, there are 4 imbeddings of $L_n$ such that the new end-rung $\overline{e}$ is a boundary of two different faces.

Similarly, if $e$ is a boundary of two different faces in an imbedding of $L_{n-1}$, then there are two ways to attach $\overline{e}$ to $e$ so that $\overline{e}$ is the boundary of two different faces, and there are also two ways to attach $\overline{e}$ to $e$ so that the two sides of $\overline{e}$ lie in the same face. Summarize these together, we get

$$d_n = 4s_{n-1} + 2d_{n-1} \qquad\qquad s_n = 2d_{n-1}$$

By the above two equalities, we have

$$d_{n+1} = 2d_n + 8d_{n-1} \qquad\qquad s_{n+1} = 2s_n + 8s_{n-1}$$

for $n \geq 2$. It is also easy to check that

$$d_1 = 2, \;\; s_1 = 2, \;\; d_2 = 12, \;\; s_2 = 4$$

Let

$$D(x) = \sum_{n=1}^{\infty} d_n x^n$$

Using the above recursive relations and the initial values, it is easy to see that

$$D(x) = 2x + 8x^2 + 2xD(x) + 8x^2 D(x)$$

so

$$1 + D(x) = \frac{1}{1 - 2x - 8x^2} = \sum_{n=0}^{\infty} \left( \frac{1}{3}(-2)^n + \frac{2}{3}4^n \right) x^n$$

This gives

$$d_n = (-1)^n \frac{2^n}{3} + 2 \cdot \frac{4^n}{3}$$

and

$$s_n = (-1)^{n-1} \frac{2^n}{3} + \frac{4^n}{3}$$

for $n \geq 1$. $\square$

**Corollary 2.2** *Let $L_n$ be the closed-end ladder of $n$ rungs. Then we have the following relations:*

$$\gamma_{avg}(L_1) = \frac{1}{2}$$

*and*

$$\gamma_{avg}(L_n) = \gamma_{avg}(L_{n-1}) + inc_{n-1} = \frac{n}{3} + \frac{1}{9}\left(1 - (-\frac{1}{2})^n\right)$$

*for $n \geq 2$, where $inc_m = \frac{1}{3}\left(1 - (-\frac{1}{2})^{m+1}\right)$, for $m \geq 1$.*

Proof. It is routine to verify that $\gamma_{avg}(L_1) = \frac{1}{2}$. Moreover, if we let $inc_m = \frac{1}{3}\left(1 - (-\frac{1}{2})^{m+1}\right)$, then it is easy to check that the sum $\gamma_{avg}(L_1) + \sum_{m=1}^{n-1} inc_m$ is equal to $\frac{n}{3} + \frac{1}{9}\left(1 - (-\frac{1}{2})^n\right)$, for $n \geq 2$. Therefore, it remains to prove that the increment of average genus from $L_{n-1}$ to $L_n$ is $inc_{n-1} = \frac{1}{3}\left(1 - (-\frac{1}{2})^n\right)$, for $n \geq 2$.

Let $e$ be an end-rung of $L_{n-1}$ (since $L_{n-1}$ is symmetric with respect to its four end-rungs, we can choose any one of them). Let $d_{n-1}$ be the number of imbeddings of $L_{n-1}$ in which $e$ is a boundary of two different faces, and let $s_{n-1}$ be the number of imbeddings of $L_{n-1}$ in which the two sides of $e$ lie in the same face. Now we attach a new edge $\overline{e}$ to $e$ to construct $L_n$ from $L_{n-1}$. For each imbedding of $L_{n-1}$, there are 4 ways to attach $\overline{e}$ to obtain an imbedding of $L_n$.

The sum of genera over all imbeddings of $L_{n-1}$ is $4^{n-1}\gamma_{avg}(L_{n-1})$. For those imbeddings of $L_{n-1}$ in which the two sides of the edge $e$ lie in the same face, adding the new edge $\overline{e}$ does not increase the genus. For each imbedding of $L_{n-1}$ in which $e$ is a boundary of two different faces, there are 2 of the 4 corresponding imbeddings of $L_n$ which increase the genus by 1. Thus the total genera increased from $L_{n-1}$ to $L_n$ is $2d_{n-1}$. Therefore the average genus increased from $L_{n-1}$ to $L_n$ is $inc_{n-1} = (2d_{n-1})/4^n$. By the formula obtained in Lemma 2.1,

$$inc_{n-1} = \frac{2d_{n-1}}{4^n} = \frac{1}{3}\left(1 - (-\frac{1}{2})^n\right)$$

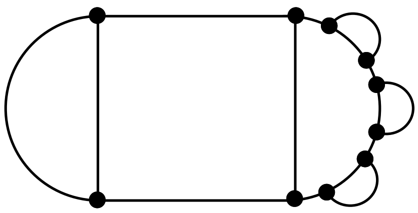

Figure 3: The 2-rung 3-ear ladder $L_{2,3}$

□

The following corollary will be useful.

**Corollary 2.3**

$$\gamma_{avg}(L_n) \geq n/3$$

We also give the results of calculations of average genus for two special classes of graphs. These results will be used later. The proofs of these results can be found in Chen and Gross [ChGr 1990c, Lemmas 5.5 and 5.6]. We recall that a graph with two vertices and $n$ edges adjoining them is called a *dipole* and is denoted $D_n$. Also, a graph with one vertex and $n$ self-loops is called a bouquet and is denoted $B_n$.

**Theorem 2.4** *For every sufficiently large number $n$, $\gamma_{avg}(D_n) \geq \frac{n}{4}$.*

**Theorem 2.5** *For every sufficiently large number $n$, $\gamma_{avg}(B_n) \geq \frac{n}{4}$.*

Consider a closed-end ladder $L_n$ with $n$ rungs. Let $e$ be an end-rung of $L_n$. Subdivide $e$ by $2k$ consecutive vertices $u_1, u_2, \cdots, u_{2k}$, then add a multiple edge (call it an *ear*) to each pair of vertices $\{u_{2i-1}, u_{2i}\}$, for $i = 1, 2, \cdots, k$. Call the resulting graph the *$n$-rung $k$-ear ladder*, denoted by $L_{n,k}$. Figure 3 depicts the 2-rung 3-ear ladder $L_{2,3}$.

We now compute the average genus of $L_{n,k}$.

**Theorem 2.6** *For any positive integer $k$,*

$$\gamma_{avg}(L_{1,k}) = 1 - \frac{1}{2^{k+1}}$$

*Moreover, for $n > 1$,*

$$\gamma_{avg}(L_{n,k}) = \left(\frac{n}{3} + \frac{7}{9} + \frac{2}{9}(-\frac{1}{2})^n\right) - \frac{1}{3 \cdot 2^{n+k}}\left(2^{n+1} + (-1)^n\right)$$

PROOF. The ladder $L_{1,k}$ is isomorphic to the necklace $N_{(k+1,0)}$ of type $(k+1, 0)$. By Theorem 1.2, the average genus of $N_{(k+1,0)}$ is $1 - \frac{1}{2^{k+1}}$. Now we work on the average genus of $L_{n,k}$ for $n > 1$. Let $e$ be a fixed end-rung of $L_n$.

By Lemma 2.1, we know that in the $4^n$ imbeddings of $L_n$, there are

$$d_n = (-1)^n \frac{2^n}{3} + 2 \cdot \frac{4^n}{3}$$

imbeddings in which the two sides of $e$ lie in different faces and

$$s_n = (-1)^{n-1} \frac{2^n}{3} + \frac{4^n}{3}$$

imbeddings in which the two sides of $e$ lie in the same face. In all those $s_n$ imbeddings in which the two sides of $e$ lie in the same face, adding the $k$ ears (in any way) does not increase the genus of the imbedding. In each of those $d_n$ imbeddings in which the two sides of $e$ lie in different faces, there are $2^k$ ways to add the $k$ ears without increasing the genus, and $4^k - 2^k$ ways to add the $k$ ears which increase the genus by 1. Thus, the total genus increased from all imbeddings of $L_n$ to all imbeddings of $L_{n,k}$ is $d_n(4^k - 2^k)$ and the average genus increased from $L_n$ to $L_{n,k}$ is

$$c_{n,k} = d_n(4^k - 2^k)/4^{n+k}$$

Replacing $d_n$ by $(-1)^n \frac{2^n}{3} + 2 \cdot \frac{4^n}{3}$, we get

$$\begin{aligned} c_{n,k} &= \frac{\frac{1}{3}[2 \cdot 4^n + (-1)^n 2^n](4^k - 2^k)}{4^{n+k}} \\ &= \frac{1}{3}\left[2 + (-1)^n \frac{1}{2^n} - \left(\frac{1}{2^{k-1}} + (-1)^n \frac{1}{2^{n+k}}\right)\right] \end{aligned}$$

Therefore, the average genus of $L_{n,k}$ is (for $n \geq 2$):

$$\begin{aligned}
\gamma_{avg}(L_{n,k}) &= \gamma_{avg}(L_n) + c_{n,k} \\
&= \frac{n}{3} + \frac{1}{9}\left(1 - (-\frac{1}{2})^n\right) \\
&\quad + \frac{1}{3}\left[2 + (-1)^n \frac{1}{2^n} - \left(\frac{1}{2^{k-1}} + (-1)^n \frac{1}{2^{n+k}}\right)\right] \\
&= \left(\frac{n}{3} + \frac{7}{9} + \frac{2}{9}(-\frac{1}{2})^n\right) - \frac{1}{3 \cdot 2^{n+k}}\left(2^{n+1} + (-1)^n\right)
\end{aligned}$$

□

**Corollary 2.7** *For any integer $n \geq 2$, the sequence $\gamma_{avg}(L_{n,k})$, for $k = 1, 2, 3, \cdots$, has the number*

$$\frac{n}{3} + \frac{7}{9} + \frac{2}{9}(-\frac{1}{2})^n$$

*as a limit point.*

**Corollary 2.8** *Any interval $[i, i+1]$ of unit length in the real line where $i \geq 1$ contains at least two limit points of values of average genus of 2-connected graphs.*

PROOF.

Let $h_1 = 1$ and $h_n = \frac{n}{3} + \frac{7}{9} + \frac{2}{9}(-\frac{1}{2})^n$ for $n \geq 2$. Then

$$h_{n+1} - h_n = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^{n+1}$$

for $n \geq 2$. We have

$$h_1 = 1, \quad h_2 = \frac{3}{2}, \quad h_3 = \frac{7}{4}, \quad h_4 = \frac{17}{8}$$

Thus, if $1 \leq i \leq \frac{7}{4}$, the corollary is correct. Otherwise, we have $i > \frac{7}{4}$. In that case, let $j$ be the largest index such that $h_j < i$. Then $h_j \geq \frac{7}{4}$ and $j \geq 3$. Now for $n \geq 3$, we have

$$h_{n+1} - h_n = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^{n+1} \leq \frac{1}{3} + \frac{2}{3 \cdot 2^4} < 0.5$$

So $h_{j+1}$ and $h_{j+2}$ must be in the interval $[i, i+1]$. □

**Corollary 2.9** *Any interval $[i, i+1]$ of unit length, where $i \geq 0.5$, contains infinitely many numbers that are values of the average genus of 2-connected graphs.*

## 3 Constructing limit points of average genus

In this section, we systematically formulate an ear-adding method to construct limit points of values of average genus for 2-connected graphs.

Let $e$ be an edge of a graph $G$. We say we *attach an open ear to (the interior of) $e$* if we insert two vertices $u$ and $v$ and double the edge between them. The two vertices $u$ and $v$ are called the *ends* of that open ear. Similarly, we *attach a closed ear to (the interior of) $e$* if we insert one new vertex $w$ and then attach a self-loop at $w$. The vertex $w$ is called the *end* of that closed ear.

We say that $r$ open ears and $s$ closed ears are attached *serially* to the edge $e$ if all ends of the ears are distinct, and no ear has an end between the two ends of an open ear. If a supergraph of $G$ is obtained by serially attaching ears to several edges of $G$, we say that the entire attachment is *serial.*

Let $G$ be a 2-connected graph. It is easy to see that no matter how we serially attach (open and/or closed) ears to the edges of $G$, the resulting graph is always 2-connected. On the other hand, we can definitely get infinitely many non-homeomorphic graphs by serially attaching ears to the graph $G$. The following theorem and corollary prove that the average genus of the graphs obtained in this way is always bounded.

**Theorem 3.1** *Let $G$ be a 2-connected graph. Let $G_+$ be a graph obtained by serially attaching ears to an edge of $G$. Then the average genus of $G_+$ is bounded by $\gamma_{avg}(G) + 1$.*

PROOF. Suppose $G_+$ is obtained by serially attaching $r$ open ears and $s$ closed ears to the edge $e$ of $G$. Consider a rotation system $R$ of the graph $G$. If the two sides of the edge $e$ are in the same face of $R$, then no matter how we attach the $r$ open ears and the $s$ closed ears to $R$, the resulting rotation system of $G_+$ has the same genus as that of $R$. On the other hand, if the two sides of the edge $e$ are in different faces of the rotation system $R$, there are $2^r 4^s$ ways to insert the $r$ open ears and the $s$ closed ears so that the genus of the resulting rotation system of $G_+$ is not increased (the genus is not increased if and only if each attached ear lies entirely in a single face); and $4^r 6^s - 2^r 4^s$ ways to attach the $r$ open ears and the $s$ closed ears so that the resulting rotation system of $G_+$ increases the genus by 1. Now suppose that $G$ has totally $N$ rotation systems (imbeddings), and that $D$ of them have the edge $e$ lying between two different faces. Then the average genus increase from $G$ to $G_+$ is

$$\frac{(4^r 6^s - 2^r 4^s) \cdot D}{4^r 6^s N}$$

which is bounded by $1 - \frac{2^{s-r}}{3^s} < 1$. $\square$

**Corollary 3.2** *Let $G$ be a 2-connected graph with $m$ edges, and let $G_+$ be a graph obtained by serially attaching ears to the edges of $G$. Then $G_+$ is 2-connected and has average genus at most $\gamma_{avg}(G) + m$.*

A question is: is it possible that to attach an ear to a graph so that the average genus does not increase at all ? The following theorem gives a negative answer to this question, and shows that attaching a single ear to an edge of a 2-connected graph always increases the average genus of the graph.

**Theorem 3.3** *Let $G$ be a 2-connected graph and let $G_+$ be a graph obtained by attaching a single ear to an edge $e$ of $G$, then*

$$\gamma_{avg}(G) < \gamma_{avg}(G_+)$$

PROOF. We are always able to construct a rotation system of $G$ such that the two sides of the edge $e$ lie on two different faces of the corresponding imbedding. For example, let $C$ be a simple cycle in $G$ that passes through the edge $e$. Then we construct a rotation system $R$ such that one side of the cycle $C$ is a face $f$ in $R$. Thus, the two sides of the edge $e$ must lie in two different faces of $R$. Now suppose that $G$ has $N$ different rotation systems. By the analysis in the proof of Theorem 3.1, the average genus increased from $G$ to $G_+$ is at least $\frac{2}{4N}$ if the attached ear is open, and at least $\frac{2}{6N}$ if the attached ear is closed. $\square$

**Corollary 3.4** *Given a 2-connected graph $G$, there are infinitely many real numbers less than $\gamma_{avg}(G)+1$ that are values of average genus of 2-connected supergraphs of $G$.*

The analysis above gives us a systematic method to construct limit points of values of average genus of 2-connected graphs.

**Theorem 3.5** *Let $G$ be a 2-connected graph. Let $G_0, G_1, G_2, \cdots,$ be a sequence of 2-connected supergraphs of $G$ obtained by serially attaching (open and/or closed) ears to $G$ such that for $i \geq 0$, $G_i$ is homeomorphic to a proper subgraph of $G_{i+1}$. Then the sequence*

$$\gamma_{avg}(G_0), \gamma_{avg}(G_1), \gamma_{avg}(G_2), \cdots, \gamma_{avg}(G_i), \cdots$$

*approaches a finite limit point.*

PROOF. By Corollary 3.2 and Theorem 3.3, the sequence of numbers

$$\gamma_{avg}(G_0), \gamma_{avg}(G_1), \gamma_{avg}(G_2), \cdots, \gamma_{avg}(G_i), \cdots$$

is strictly increasing with an upper bound $\gamma_{avg}(G_0) + m$. Thus the sequence must approach a finite limit point. $\square$

## 4 Uniqueness of the limit-point construction

The method we used in the previous sections is in fact a generalization of a method derived by Klein (see [GrKlRi 1989]) who first pointed out that all necklaces have average genus less than 1. In a necklace, the basic simple cycle serves as a "frame" and all other edges are just ears serially attached to the frame. Similarly, in a $n$-rung $k$-ear closed-end ladder $L_{n,k}$, the subgraph $L_n$ is a frame and all other edges are ears serially attached to the frame. We have also proved in Section 3 that if we start with an arbitrary 2-connected graph $G$ as a frame, and serially attach ears to the edges of $G$, we will obtain a sequence of non-homeomorphic graphs whose average genus values approach a finite limit point. In this section, we prove that this is the only way to make limit points of values of average genus for homeomorphically nested sequence of 2-connected graphs.

We first present a few lemmas concerned with the following problem: given a 2-connected subgraph $H$ of a 2-connected graph $G$, suppose we know the relationship between the values of the average genus of $G$ and $H$. What can we say about the relationship between the structures of the two graphs?

The beautiful theory of bridges and attachments, as presented by Tutte [Tutte 1966] in his classic study of graph connectivity, is needed for our discussion. We slightly modify the terminology used in [Tutte 1984].

Let $H$ be a subgraph of a connected graph $G$. Denote by $G - H$ the graph obtained from $G$ by first deleting all edges in $H$ and then deleting all the isolated vertices. An *attachment* of $H$ in $G$ is a vertex that lies both in $H$ and in $G - H$.

If some edge of $G - H$ has both its ends in the subgraph $H$, then it is called a *trivial bridge* of $G - H$. Let $W$ be the (possibly not connected) subgraph of $G$ obtained by deleting all vertices (and the edges incident on them) in $V(H)$ from $G$ and let $C$ be a connected component of $W$. Let $B$ be the subgraph of $G$ obtained from $C$ by adjoining to it each edge of $G$ with one end in $C$ and one in $H$, together with the end-vertex of that edge in $H$. Then $B$ is called a *nontrivial bridge* of $G - H$. A subgraph $B$ of $G$ is called a *bridge* of $G - H$ if $B$ is either a trivial bridge or a nontrivial bridge.

Let $B$ be a bridge of $G - H$. Then $B$ is a connected subgraph of $G$. Moreover, a nontrivial bridge with some of its attachments deleted is also a connected subgraph of $G$. If a vertex $v$ belongs to two distinct bridges of $G - H$, then $v$ must be an attachment of the subgraph $H$ in $G$. These follow directly from the definitions. For further discussion of bridges and attachments, see Tutte [Tutte 1984]. We will also investigate some special properties of bridges and attachments of 2-connected graphs below.

A graph is called *smooth* if it does not contain 2-valent vertices.

**Lemma 4.1** *Let $H$ be a 2-connected subgraph of a smooth 2-connected graph $G$ such that*

$$\gamma_{avg}(G) - \gamma_{avg}(H) < \frac{1}{3}$$

*Then all bridges of $G - H$ are trivial bridges.*

PROOF. We shall suppose that $B$ is a non-trivial bridge of $G - H$ and seek a contradiction.

Since $G$ is 2-connected, the bridge $B$ has at least two distinct attachments $v_1$ and $v_2$. Let $p_1$ be a simple path in $B$ that joins the two vertices $v_1$ and $v_2$. The path $p_1$ cannot be a single edge, since $p_1$ is putatively contained in the non-trivial bridge $B$. Therefore, we must be able to find another simple path $p_2$ in $B$ from an interior vertex $v_3$ of the path $p_1$ to another vertex $v_4$, which is either another interior vertex of $p_1$ or an attachment of the bridge $B$, such that no interior vertex of $p_2$ lies on $p_1$. The vertex $v_4$ may be identical with either $v_1$ or $v_2$, or the vertex $v_4$ may be identical with the vertex $v_3$.

Now we calculate the increment of the average genus obtained when we attach these two paths $p_1$ and $p_2$ to the graph $H$. Denote by $H_+$ the subgraph of $G$ that consists of the graph $H$ plus the path $p_1$, and denote by $H_{++}$ the subgraph of $G$ that consists of the graph $H$ plus the two paths $p_1$ and $p_2$. Our objective is to show that $\gamma_{avg}(H_{++}) - \gamma_{avg}(H) \geq 1/3$.

Suppose that the valences in the graph $H$ of the vertices $v_1$ and $v_2$ are $d_1$ and $d_2$, respectively, and suppose that the valences in the graph $H$ of the vertices $v_3$ and $v_4$ are $d_3$ and $d_4$, respectively.

Given a rotation system $R_H$ of the graph $H$, we can get $d_1 \cdot d_2$ different rotation systems of the graph $H_+$ by attaching the end-vertex $v_1$ of the path $p_1$ to the $d_1$ different corners of the vertex $v_1$ in the rotation system $R_H$, and attaching the end-vertex $v_2$ of the path $p_1$ to the $d_2$ different corners of the vertex $v_2$ in the rotation system $R_H$. Similarly, given a rotation system $R_{H+}$ of the graph $H_+$, we can get $d$ different rotation systems of the graph $H_{++}$ by attaching the path $p_2$ to $R_{H+}$, where $d = d_3 \cdot d_4 = 2d_4$ if $v_3$ and $v_4$ are distinct, since $v_3$ is an interior vertex of $p_1$ and has valence 2 in the graph $H_+$, and $d = 2 \cdot 3 = 6$ if $v_3$ and $v_4$ are identical, in which case, the path $p_2$ is a self-loop at the middle of the path $p_1$.

Now suppose that the path $p_1$ is attached to the rotation system $R_H$ of the graph $H$ in such a way that the vertices $v_1$ and $v_2$ of $p_1$ are attached to the corners $C_1$ and the corner $C_2$, respectively. There are two different cases.

CASE 1. The two corners $C_1$ and $C_2$ belong to different faces of the rotation system $R_H$.

Then the path $p_1$ merges the two different faces of the rotation system $R_H$, and the resulting rotation system $R_{H+}$ increases the genus by 1 from the rotation system $R_H$. That is,

$$\gamma(R_{H+}) = \gamma(R_H) + 1$$

where $\gamma(R)$ denotes the genus of the rotation system $R$. Hence, no matter how we attach the path $p_2$ to the rotation system $R_{H+}$ (there are $d$ different ways, where $d$ is defined as above), the resulting rotation system $R_{H++}$ of the graph $H_{++}$ has genus at least one larger than that of the rotation system $R_H$. (end of Case 1)

CASE 2. The two corners $C_1$ and $C_2$ belong to the same face of the rotation system $R_H$.

Then attaching the path $p_1$ to $R_H$ does not increase the genus. However, the two sides of the path $p_1$ now lie in different faces of the resulting rotation system $R_{H+}$ of the graph $H_+$. Now we attach the path $p_2$ to $R_{H+}$. If $v_3 \neq v_4$, then for each corner $C$ of the $d_4$ corners of the vertex $v_4$ in the rotation system $R_{H+}$, at least one of the two corners of the vertex $v_3$ in $R_{H+}$ is contained in a face that does not contains the corner $C$. Therefore, there are at least $d_4 = d/2$ ways to attach the path $p_2$ to the rotation system $R_{H+}$ so that the path $p_2$ merges two different faces and increases the genus by 1. On the other hand, if $v_3 = v_4$, then in the 6 different ways to attach the self-loop $p_2$ to the vertex $v_3$ on the path $p_1$, exactly two of them merge two different faces of $R_{H+}$ and thereby increase the genus by 1. (end of Case 2)

Now suppose that in the $d_1 \cdot d_2$ pairs of corners $(C_1, C_2)$, where $C_1$ is a corner of the vertex $v_1$ in the rotation system $R_H$, and $C_2$ is a corner of the vertex $v_2$ in the rotation system $R_H$, there are $D$ of them such that the corners $C_1$ and $C_2$ belong to different faces of $R_H$, and $S$ of them such that the corners $C_1$ and $C_2$ belong to the same face of $R_H$. As we have discussed above, there are at least

$$d \cdot D + \frac{d}{3} \cdot S \geq \frac{d}{3}(D + S) = \frac{1}{3} d_1 d_2 d$$

ways to attach the paths $p_1$ and $p_2$ to the rotation system $R_H$ that increase the genus by at least 1.

If we let $N$ be the number of rotation systems of the graph $H$, then the average genus increased from the graph $H$ to the graph $H_{++}$ is at least

$$(\frac{1}{3}d_1 d_2 d \cdot N)/(d_1 d_2 d \cdot N) = \frac{1}{3}$$

i.e., $\gamma_{avg}(H_{++}) \geq \gamma_{avg}(H) + \frac{1}{3}$, where $d_1 d_2 d \cdot N$ is the number of rotation systems of the graph $H_{++}$. Since $H_{++}$ is a subgraph of $G$, by Theorem 1.1 we have

$$\gamma_{avg}(H) + \frac{1}{3} \leq \gamma_{avg}(H_{++}) \leq \gamma_{avg}(G) < \gamma_{avg}(H) + \frac{1}{3}$$

This contradiction completes the proof of the theorem. □

Therefore, if $G$ is a 2-connected graph and $H$ is a 2-connected subgraph of $G$, and $\gamma_{avg}(G) - \gamma_{avg}(H) < 1/3$, then the graph $G$ can be obtained from $H$ by attaching edges to the graph $H$. We now investigate the relationship between these attached edges.

**Lemma 4.2** *Let $H$ be a 2-connected subgraph of a smooth 2-connected graph $G$, such that*

$$\gamma_{avg}(G) - \gamma_{avg}(H) < \frac{1}{3}$$

*Then no two bridges of $G - H$ can share as their common attachment a vertex that is 2-valent in $H$.*

PROOF. By Lemma 4.1, all bridges of $G - H$ are trivial bridges. Now suppose that two bridges $B$ and $B'$ share as their common attachment a vertex $a$ that is 2-valent in $H$. There are three possible cases:

1. Neither of the bridges $B$ and $B'$ is a self-loop,

2. Exactly one of the bridges $B$ and $B'$ is a self-loop,

3. Both the bridges $B$ and $B'$ are self-loops.

To complete the proof, it suffices to show that in each of these three cases, adding the bridges $B$ and $B'$ to the subgraph $H$ raises its average genus by at least 1/3. The considerations are similar for all three cases, and we presently provide details for Case 3.

Case 3. We shall consider all the ways to extend an arbitrary imbedding of $H$ by adding first $B$ and then $B'$, and we shall see that at least one-third of them increase the genus. If the vertex $a$ lies on two faces of the imbedding of $H$, then two of the six ways to attach the self-loop $B$ already increase the genus, thereby assuring that the genus increases for at least one-third of the ways to add both $B$ and $B'$. Therefore, let us suppose that the vertex $a$ lies on only one face of the imbedding of $H$, so that none of the six ways of adding self-loop $B$ increases the genus. In four of these six imbeddings of $H + B$, the self-loop $B$ bounds a monogon, and six of the 20 ways to add self-loop $B'$ increase the genus. In the other two imbeddings of $H + B$, eight of the 20 ways to add self-loop $B'$ increase the genus. Thus, of the 120 ways to add both $B$ and $B'$, 40 (i.e., $4 \cdot 6 + 2 \cdot 8$) increase the genus. $\square$

Let $G$ be a graph. By a *suspended chain* $C$ of $G$ we mean a simple open path in $G$ such that all interior vertices of $C$ have valence 2 in $G$. By this definition, a edge that is not a self-loop is a suspended chain. A suspended chain $C$ of $G$ is *maximal* if its two end-vertices have valence greater than 2 in $G$.

**Lemma 4.3** *Let $H$ be a 2-connected subgraph of a smooth 2-connected graph $G$ such that*

$$\gamma_{avg}(G) - \gamma_{avg}(H) < \frac{1}{3}$$

*Let $B$ be a bridge of $G - H$ such that $B$ has two distinct attachments $a_1$ and $a_2$ on a suspended chain $C$ of the graph $H$. Then the subchain $C'$ adjoining $a_1$ and $a_2$ on the chain $C$ must be a single edge of $G$.*

PROOF. By Lemma 4.1, $B$ is a trivial bridge. Consider the graph $H' = H - C' + B$, which is obtained by deleting the subchain $C'$ adjoining $a_1$ and $a_2$ on the chain $C$, then adding the edge $B$ to the resulting graph. The graph $H'$ is obviously homeomorphic to the graph $H$. Therefore,

$$\gamma_{avg}(G) - \gamma_{avg}(H') < \frac{1}{3}$$

By Lemma 4.1 again, all bridges of $G - H'$ are trivial bridges. In particular, the subchain $C'$ must be contained in some trivial bridge of $G - H'$. That is, $C'$ itself must be a single edge of $G$. □

Let $G$ be a graph and let $H$ be a subgraph of $G$. Suppose that $C$ and $C'$ are two distinct maximal suspended chains in $H$. We say that a trivial bridge $B$ of $G - H$ *links the interiors* of $C$ and $C'$ if one attachment of $B$ is an interior vertex of $C$ and the other attachment of $B$ an interior vertex of $C'$.

Let $B^{(1)}$, $B^{(2)}$, $\cdots$, $B^{(r)}$ be $r$ trivial bridges of $G - H$, such that each bridge $B^{(i)}$ has two attachments $a_i$ and $b_i$, for $i = 1, 2, \cdots, r$. We say that the bridges $B^{(1)}$, $B^{(2)}$, $\cdots$, $B^{(r)}$ *link the interiors of* $C$ *and* $C'$ *in order* if each $B^{(i)}$ links the interiors of $C$ and $C'$, and if we can travel the attachments on the suspended chains $C$ and $C'$ in the respective orderings $a_1$, $a_2$, $\cdots$, $a_r$, and $b_1$, $b_2$, $\cdots$, $b_r$. We observe that there are two ways to link the interiors of chains $C$ and $C'$ in order, depending on the direction in which $C$ and $C'$ are traversed.

**Lemma 4.4** *Let $H$ be a 2-connected subgraph of a smooth 2-connected graph $G$ such that*

$$\gamma_{avg}(G) - \gamma_{avg}(H) < \frac{1}{4}$$

*and that three trivial bridges of $G - H$ link the interiors of two distinct maximal suspended chains $C$ and $C'$ of $H$. Then they must link the interiors in order.*

PROOF. Suppose that for $i = 1, 2, 3$, $B^{(i)}$ is a trivial bridge that links the interiors of the maximal suspended chains $C$ and $C'$, with an attachment $a_i$ on $C$ and an attachment $b_i$ on $C'$. By Lemma 4.2, all these attachments $a_i$'s and $b_i$'s are distinct. Without loss of generality, we suppose that on the suspended chain $C$, the vertex $a_2$ is between the vertices $a_1$ and $a_3$.

Suppose that the lemma is not true and that $b_2$ is not between $b_1$ and $b_3$ on the chain $C'$. So either $b_1$ is between $b_2$ and $b_3$ on the chain $C'$, or $b_3$ is between $b_1$ and $b_2$ on the chain $C'$. Since these two cases are symmetric, we can assume, without loss of generality, that $b_1$ is between $b_2$ and $b_3$ on the chain $C'$.

Given a rotation system $R$ of $H$, we first attach the edge $B^{(1)}$. If $B^{(1)}$ merges two different faces of $R$, then any way of attaching $B^{(2)}$ and $B^{(3)}$ yields a rotation system of the graph $H+B^{(1)}+B^{(2)}+B^{(3)}$ with genus at least one larger than $R$. If $B^{(1)}$ is attached to $R$ in such a way that $B^{(1)}$ is a chord of a face of $R$, then since $b_1$ lies between $b_2$ and $b_3$ on the chain $C'$, it follows that some corner at $b_2$ and some corner at $b_3$ belong to two different faces of the resulting rotation system $R_+$ of the graph $H + B^{(1)}$. Since attachments $a_2$ and $a_3$ both lie past attachment $a_1$ on the chain $C$, some corner of $a_2$ and some corner of $a_3$ belong to the same face of the rotation system $R_+$. Therefore, in the rotation system $R_+$, either $a_2$ has a corner belonging to a face that does not contain a corner of $b_2$, or $a_3$ has a corner belonging to a face that does not contain a corner of $b_3$. Without loss of generality, suppose $a_2$ has a corner belonging to a face that does not contain a corner of $b_2$. Then among the four ways to attach $B^{(2)}$, at least one of them increases the genus by one. Finally we attach $B^{(3)}$ arbitrarily. Now an analysis similar to that in the proof of Lemma 4.1 gives the conclusion that the average genus increased from the graph $H$ to the graph $H + B^{(1)} + B^{(2)} + B^{(3)}$ is at least

1/4. So we have

$$\gamma_{avg}(H + B^{(1)} + B^{(2)} + B^{(3)}) \geq \gamma_{avg}(H) + \frac{1}{4} > \gamma_{avg}(G)$$

This contradicts Theorem 1.1, since the graph $H + B^{(1)} + B^{(2)} + B^{(3)}$ is a subgraph of the graph $G$. $\square$

**Corollary 4.5** *Let $H$ be a 2-connected subgraph of a smooth 2-connected graph $G$ such that*

$$\gamma_{avg}(G) - \gamma_{avg}(H) < \frac{1}{4}$$

*and that $r$ trivial bridges of $G - H$ link the interiors of two distinct maximal suspended chains $C$ and $C'$ of $H$. Then they must link the interiors in order.*

Now we are ready to prove that our construction in Section 3 is essentially the only method for constructing limit points of values of average genus of "homeomorphically nested" 2-connected graphs.

We call a sequence $G_1, G_2, G_3, \cdots$, of graphs a *strictly monotone sequence* if no pair of graphs in the sequence are homeomorphic and each $G_j$ is homeomorphic to a subgraph of $G_{j+1}$, for all $j \geq 1$.

**Theorem 4.6** *Let $G_1, G_2, G_3, \cdots$, be a strictly monotone sequence of 2-connected graphs such that the values of average genera of the graphs approach a finite limit point. Then there exists an index $I$ such that all but a finite number of graphs in the sequence can be obtained by serially attaching ears to $G_I$.*

Proof. Suppose that the average genus of the graphs in the sequence

$$\Gamma : \quad G_1, G_2, \cdots, G_i, \cdots$$

approaches a limit point $A < \infty$. Since the sequence is strictly monotone, it follows from Theorem 1.1 that the average genus of any graph in the sequence is no larger than $A$.

Let $J$ be an index in the sequence such that

$$A - \gamma_{avg}(G_J) < \frac{1}{4}$$

Then the graph $G_J$ is homeomorphic to a subgraph of any graph $G_i$ in the sequence with $i \geq J$, and

$$\gamma_{avg}(G_i) - \gamma_{avg}(G_J) < \frac{1}{4}$$

Without loss of generality, we suppose that for $i \geq J$, each $G_i$ is smooth and $G_J$ is a subgraph of $G_i$. By Lemma 4.1, the graph $G_i$ can be obtained by attaching a set of trivial bridges to the graph $G_J$ such that the endpoints of every bridge are in the graph $G_J$. A bridge of $G_i - G_J$ lies in one of the following seven possible classes (a trivial bridge is called an *open bridge* if it is not a self-loop):

1. The bridge $e$ is a self-loop and is attached to a vertex of valence greater than 2 in $G_J$.

2. The bridge $e$ is an open edge with its two endpoints attached to two vertices both of valence greater than 2 in $G_J$.

3. The bridge $e$ is an open bridge such that one of its endpoint is attached to a vertex $v$ of valence greater than 2 in $G_J$, and the other endpoint of it is attached to an interior vertex of a maximal suspended chain $C$ in $G_J$ such that $v$ is an end-vertex of $C$.

4. The bridge $e$ is an open bridge such that one of its endpoint is attached to a vertex $v$ of valence greater than 2 in $G_J$, and the other endpoint is attached to an interior vertex of a maximal suspended chain $C$ in $G_J$ such that $v$ is not an end-vertex of $C$.

5. The bridge $e$ is an open bridge and links the interiors of two different maximal suspended chains in $G_J$.

6. The bridge $e$ is an open bridge such that its two endpoints are attached to two different interior vertices of a maximal suspended chain in $G_J$,
7. The bridge $e$ is a self-loop and is attached to an interior vertex of a maximal suspended chain in $G_J$.

Suppose that the graph $G_J$ has $n$ vertices of valence greater than 2 and $m$ maximal suspended chains. We now consider how many edges in each class above can be attached to the graph $G_J$ to get a graph $G_i$ in the given sequence $\Gamma$. Let $\alpha = \lceil A \rceil$.

At each vertex $v$ of valence greater than 2 in $G_J$, at most $4\alpha$ self-loops can be attached, since attaching $h > 4\alpha$ self-loops to $v$ yields a supergraph of the bouquet $B_h$, with average genus at least $h/4 > \alpha$, by Theorem 2.5. Then by Theorem 1.1, the resulting graph would have average genus greater than $\alpha$, and be not in the given sequence $\Gamma$. Therefore, at most $4\alpha n$ edges in Class 1 can be attached to the graph $G_J$.

For each pair of vertices of valence greater than 2 in $G_J$, at most $4\alpha$ multiple edges can be attached, since attaching $h > 4\alpha$ multiple edges to a pair of vertices yields a supergraph of the dipole $D_h$, with average genus at least $h/4 > \alpha$, by Theorem 2.4. Therefore, at most $4\alpha n^2$ edges in Class 2 can be attached to the graph $G_J$.

For each vertex $v$ of valence greater than 2 in $G_J$ and each maximal suspended chain $C$ in $G_J$ with $v$ as one of its end-vertex, at most one trivial bridge $e$ can be attached so that one endpoint of $e$ is on $v$ and the other endpoint is attached to an interior vertex of $C$, by Lemma 4.2 and Lemma 4.3. Therefore, at most $2m$ edges in Class 3 can be attached to the graph $G_J$.

For each vertex $v$ of valence greater than 2 in $G_J$ and each maximal suspended chain $C$ in $G_J$ such that $v$ is not an end-vertex of $C$, at most $8\alpha+1$ trivial bridges can be attached such that one endpoint of $e$ is on $v$ and the other endpoint is attached to an interior vertex of $C$. This is because for two such bridges, the two attachments on the chain $C$ must be distinct, by Lemma 4.2. Thus after we attach $h > 8\alpha + 1$ such bridges, the graph consisting of the chain $C$, the vertex $v$, and these $h$ trivial bridges contains a subgraph which is homeomorphic to the bouquet $B_{4\alpha+1}$, which has average genus greater than $\alpha$. Therefore, at most $nm(8\alpha + 1)$ edges in Class 4 can be attached to the graph $G_J$.

For each pair $(C, C')$ of two distinct maximal suspended chains of $G_J$, if we attach $3\alpha + 3$ edges that link the interiors of $C$ and $C'$, then by Corollary 4.5, these $3\alpha + 3$ edges link the interiors of $C$ and $C'$ in order. Consequently, the resulting graph contains a subgraph that is homeomorphic to the $(3\alpha+1)$-rung closed-end ladder $L_{3\alpha+1}$, that has average genus greater than $\alpha$, by Corollary 2.3. Therefore, at most $m^2(3\alpha + 2)$ edges in Class 5 can be attached to the graph $G_J$.

Altogether, therefore, at most

$$4\alpha n + 4\alpha n^2 + 2m + nm(8\alpha + 1) + m^2(3\alpha + 2) \leq 24\alpha n^2 m^2$$

edges in Classes 1-5 can be attached to $G_J$ to construct a graph in the sequence

$$\Gamma : \quad G_1, G_2, \cdots, G_i, \cdots$$

Since the number $24\alpha n^2 m^2$ is independent of the choice of the graph $G_i$ in the above sequence, there must be an index $I \geq J$ such that for all graphs $G_i$ with $i \geq I$, only the bridges in Class 6 and/or Class 7 can be attached. By Lemma 4.2 and Lemma 4.3, an bridge in Class 6 or Class 7 must be completely disjoint from any other attached bridges, i.e., it does not have a common attachment with any other attached bridges, and no other attached bridge can have an attachment between the two attachments of a bridge in Class 6. Therefore, to obtain a graph $G_i$ in the sequence $\Gamma$ with $i \geq I$, each bridge attached to the graph $G_I$ must be an ear attached to an edge of the graph $G_I$, and the attachment of these ears to the graph $G_I$ must be serial. That is, the graph $G_i$ with $i \geq I$ can be obtained by serially attaching ears to the graph $G_I$.

This completes the proof. $\square$

Thus given a strictly monotone sequence of 2-connected graphs such that the values of the average genus of the graphs approach a finite limit point, then we can always find a 2-connected graph $G_I$, so that all but a finite number of graphs in the sequence can be obtained by serially attaching ears to the graph $G_I$. This shows that the method we adopted in Section 3 is essentially the only way to construct limit points of values of average genus of a strictly monotone sequence of 2-connected graphs.

## 5 Remarks and open problems

We have investigated the limit points of values of average genus of 2-connected graphs. We generalized Klein's construction of necklaces and developed a systematic method for constructing limit points of values of average genus for 2-connected graphs, by adding ears. We have proved that our method is essentially the only way to construct limit points of values of average genus for strictly monotone sequences of 2-connected graphs.

The study of average genus of graph may give us a way to investigate the isomorphism type properties of graphs, and has a potential applicability to graph isomorphism testing problem. The average genus of a graph can be estimated by sampling. By our results in part (I) of this paper [ChGr 1990c], at least for 2-connected simplicial graphs and for 3-connected graphs, two non-homeomorphic graphs have a negligible chance to share the same real number as their average genus. In [ChGr 1990b], we have proven that aside from eight exceptions, a graph of average genus less than 1 must be a necklace. This implies a linear-time algorithm for graph homeomorphism testing for graphs of average genus less than 1 [ChGr 1990a]. This result can be extended to the class of all 2-connected graphs whose average genus is bounded by a fixed constant [Chen 1990], by the technique used in the current paper to study of the limit points of values of average genus of 2-connected graphs.

Now we offer a few open problems for further research.

Strictly monotone sequences of graphs are not the only possible sequence of graphs. Perhaps the two consecutive graphs in a sequence are entirely unrelated. When the values of average genus of the graphs in such a sequence approach a finite limit point, what can we say about the structures of the graphs?

A limit point of values of average genus of graphs in a strictly monotone sequence is always an upper limit point of the values. Do there exist lower limit points of values of average genus of 2-connected graphs? This seems to require a "magic" method to construct the sequence of graphs so that the values of the average genus of the graphs in the sequence approach a lower limit point. Since a later graph in the sequence cannot be based on the earlier graphs in the sequence: we need infinitely many graphs, and the graphs in the sequence should get smaller and smaller.

Finally, we would like to know the density of limit points of values of average genus of 2-connected graphs. How many limit points of average genus of 2-connected graphs can we have in a finite interval on the real line? We have proved that in the interval $[0,1]$, the number 1 is the only such a limit point [ChGr 1990b]. And in Section 2, we proved that in each interval $[i, i+1]$, for $i \geq 1$, there are at least 2 such limit points. However, these do not exclude the possibility that in some finite interval on the real line, there are infinitely many limit points for values of average genus of 2-connected graphs.

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