Limit Points for Average Genus (I) : 3-Connected and 2-Connected Simplicial Graphs

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Abstract

It is demonstrated that a given value of average genus is shared by at most finitely many 2-connected simplicial graphs and by at most finitely many 3-connected graphs. Moreover, the distribution of values of average genus is sparse, in the following sense: within any finite real interval, there are at most finitely many different numbers that are values of average genus for 2-connected simplicial graphs or for 3-connected graphs. Thus, there are no limit points for the values of average genus of graphs in these classes. The potential applicability of these results to graph isomorphism testing is considered.

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1 Introduction

In this paper, one of a series, a topological invariant of recent interest, the *average genus* of graphs, is investigated. By the average genus of a graph G, we mean the average value of the genus of the imbedding surface, taken over all orientable imbeddings of G. This value is evidently a rational number, and it is clearly an invariant of the homeomorphism type of a graph.

Average genus of individual graphs is in the Gross-Furst hierarchy [GrFu 1987] and studied by Gross, Klein and Rieper [GrKlRi 1989]. (Stahl [Stahl 1983] studied the related problem of average genus of a class of graphs with the same number of edges.) In [GrKlRi 1989] it was proved that the number 1 is a limit point of the set of possible values for average genus and that the complete graph K_4 is the only 3-connected graph whose average genus is less than one. At the end of that paper, it was asked to characterize the real numbers that are values of average genus of graphs and to investigate the limit points of this kind of real numbers.

Another motivation for this paper is the pursuit of an invariant of graphs that can distinguish non-homeomorphic graphs efficiently. For this purpose, the quality of an invariant is measured against two ideal criteria: (1) it should be sufficient to distinguish non-homeomorphic graphs (i.e., a *complete topological invariant* of graphs), and (2) it should be feasibly computable. One well-known topological invariant of graphs is the minimum genus. However, the minimum genus is surely not an adequate invariant for isomorphism testing. In fact, for each non-negative integer g, there are infinitely many graphs sharing g as their minimum genus (for example, there are infinitely many planar graphs). Moreover, it is apparently quite time-consuming to compute the minimum genus of a graph [Thomassen 1989]. We view the two criteria as goals to be achieved.

We will indicate that the average genus of graphs is a reasonable start on a topological invariant for homeomorphism-type testing by showing several interesting properties of the values of average genus of 2-connected simplicial graphs and of 3-connected graphs. We shall extend a classical result of Whitney [Whitney 1932] and use our extensions to prove that a given value of average genus is shared by at most finitely many 2-connected simplicial graphs and by at most finitely many 3-connected graphs. Moreover, we prove that the distribution of values of average genus is sparse, in the following sense: within any finite real interval, there are at most finitely many different numbers that are values of average genus for 2-connected simplicial graphs or for 3-connected graphs. Consequently, there are no limit points for values of average genus of 2-connected simplicial graphs or 3-connected graphs.

The outline of this paper is as follows. In Section 2 we introduce the basic results of theory of imbedding distribution of a graph, the concept of linear synthesis, a classical result of Whitney and our extensions. We then prove in Section 3 our main results for 2-connected, 3-regular, simplicial graphs. Our results on average genus are then extended to 2-connected simplicial graphs in Section 4, and to 3-connected graphs in Section 5. In Section 6, Some open problems are proposed and the potential applicability of our results to graph homeomorphism and graph isomorphism testings is discussed.

2 Linear synthesis and other preliminaries

It is assumed that the reader is somewhat familiar with topological graph theory. We briefly review the fundamentals. For further description, see Gross and Tucker [GrTu 1987] or White [White 1984].

A *graph* may have multiple adjacencies or self-adjacencies. An *imbedding* must have the "cellularity property" that the interior of every region is simply connected. The closed orientable surface of genus n is denoted S_n.

A *rotation* at a vertex v is a cyclic permutation of the edge-ends incident on v. Thus, a d-valent vertex admits $(d-1)!$ rotations. A list of rotations, one for each vertex of the graph, is called a *rotation system*.

An imbedding of a graph G in an orientable surface induces a rotation system, as follows: the rotation at vertex v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in a orientation-preserving tour around v. Conversely, by the Heffter-Edmonds principle, every rotation system induces a unique imbedding of G into an orientable surface. The bijectivity of this correspondence implies that the number of different ways to imbed a graph of valence sequence $d_1, \cdots, d_n$ into a closed, orientable surface is

$$\prod_{i=1}^{n}(d_i - 1)!$$

For any graph G, if the number of imbeddings in the surface S_k is denoted g_k, then the sequence

$$g_0, g_1, g_2, \cdots$$

is called the *genus distribution* of G. Genus distributions for special families of graphs have been calculated by Furst, Gross, and Statman [FuGrSt 1989] and by Gross, Robbins, and Tucker [GrRoTu 1989], by blending topology and combinatorial enumeration.

The *average genus* of G is defined to be the value

$$\gamma_{avg}(G) = \frac{\sum_{i=0} i \cdot g_i}{\sum_{i=0} g_i}$$

Now we introduce the concept of linear synthesis of graphs, which originates from a classical work by Whitney [Whitney 1932], and state a form of Whitney's theorem.

Definition A *linear (iso-)synthesis* of a graph G from a subgraph G_0 is a sequence of increasing 2-connected subgraphs G_0, G_1, $\cdots$, G_{r-1}, $G_r \cong G$, such that, for $i = 1, \cdots, r$, the graph complement $G_i - G_{i-1}$ is a path (without any repeated vertices in its interior) with its two end-vertices contained in G_{i-1}. If each such path is open (so that its initial and terminating vertices are distinct from each other and from the interior vertices), then the increasing sequence is called an *open* linear synthesis. The paths $G_i - G_{i-1}$, $i = 1, \cdots, r$, are called the *paths* of the linear synthesis and r the *length* of the linear synthesis.

It should be noticed that adding a path to a graph G increases the cycle rank $\beta(G)$ of the graph by 1. Therefore, the length of a linear synthesis of a graph from a simple cycle equals $\beta(G) - 1$, so it is a topological invariant under graph homeomorphism.

Whitney established the following interesting relationship between graph connectivity and linear synthesis of graphs.

Theorem 2.1 *([Whitney 1932])*[3] *A graph G without self-loops has an open linear synthesis from a simple cycle if and only if G is 2-connected.*

A variation called a linear *homeo-synthesis* is just like an iso-synthesis, except that we allow the members of the sequence of graphs to be merely homeomorphic to subgraphs of G. This variation absorbs the inverse operations of subdividing an edge and smoothing a 2-valent vertex.

One may readily perceive that many variations of graph synthesis could be useful and that one would study the properties of the subgraphs in the

[3] In his seminal work [Whitney 1932], Whitney neither mentioned this result nor introduced the terminology "linear synthesis" explicitly. What he proved is that any 2-connected graph contains a path whose removal results in a 2-connected graph. Linear synthesis of graphs has recently found applications in theoretical computer science where the terminology "ear decomposition" is used instead of "linear synthesis" [MiRa 1987].

corresponding increasing sequences. This viewpoint enabled us to develop the following two variations of Whitney's theorem [ChGr 1990] (see also [Chen 1990]):

Theorem 2.2 *A 2-connected, 3-regular simplicial graph has an open linear homeo-synthesis from the complete graph K_4.*

Theorem 2.3 *Let G be a 2-connected, 3-regular simplicial graph. Then there is an open linear homeo-synthesis of G from a simple cycle such that for any three consecutive graphs in the sequence, at least one of them is simplicial.*

The following two lemmas will be used in our proofs.

Lemma 2.4 *The average genus of a graph G is not less than the average genus of any of its subgraphs.*

PROOF. Since a bar-amalgamation of a tree onto any graph does not change the average genus, it suffices to consider the case of a spanning subgraph G_1. Indeed, without loss of generality, we suppose that G is obtained by adding an edge e to G_1. Each rotation system R_1 of G_1 can be extended to c rotation systems of G, where c is a fixed constant (if the two end points of e are connected to two vertices v_1 and v_2 of G_1, of valence d_1 and d_2, respectively, then $c = d_1 \cdot d_2$ if v_1 and v_2 are different and $c = d_1(d_1+1)$ if v_1 and v_2 are identical). Each such a rotation system of G has genus at least as large as the genus of rotation system R_1 of G_1. Therefore, the average genus of G is at least as large as

$$\frac{c \cdot \gamma_{avg}(G_1) \cdot N_{G_1}}{c \cdot N_{G_1}} = \gamma_{avg}(G_1)$$

where N_{G_1} is the number of rotation systems of G_1. □

Lemma 2.5 *The maximum genus of a graph G is at least as large as the maximum genus of any of its subgraphs.*

PROOF. Again we suppose that G is obtained by adding an edge e to a subgraph G_1. Starting from a rotation system R_1 of G_1 that corresponds to a maximum genus imbedding of G_1. Adding e to R_1 in arbitrary way results in a rotation system of G with genus at least as large as the genus of R_1. □

A few remarks on our terminology. Since we are examining topological properties of graphs, we always ignore all 2-valent vertices. In other words, we always "smooth" all 2-valent vertices. If any intermediate graph G contains 2-valent vertices during our discussion, we will replace it by a homeomorphic graph G' obtained by smoothing all 2-valent vertices of G. We will abuse the terminology by referring to the graph G' as the graph G whenever there is no confusion.

A graph is *simplicial* if it contains neither multiple edges nor self-loops. A graph is *k-connected* if removing any $k-1$ vertices from the graph does not disconnect the graph. A graph is *k-regular* if every vertex in the graph has valence k. By our convention above, subdividing edges by introducing 2-valent vertices does not change these properties of a graph. Whenever we talk about graphs with some property (simplicialness, connectivity, or regularity), we really mean "the graphs that have this property after smoothing all 2-valent vertices".

3 On the average genus of 2-connected 3-regular simplicial graphs

In this section we prove that no real number can be a limit point of the values of the average genus of 2-connected 3-regular simplicial graphs.

Lemma 3.1 *Let G be a 2-connected, 3-regular simplicial graph with $3r$ edges, and let k be any integer such that $1 \leq k \leq r$. Then there is a 2-connected, 3-regular simplicial graph H, homeomorphic to a subgraph of G, such that H has $3k$, $3k+3$, or $3k+6$ edges and that G has an open linear homeo-synthesis from H.*

PROOF. By Theorem 2.3, G has an open linear homeo-synthesis G_0, G_1, $\cdots$, G_{r-1}, $G_r \cong G$ from a simple cycle G_0. Since G is 2-connected and 3-regular, the graph G_i, $i = 1, \cdots, r$, must be 2-connected and 3-regular (remember that we smooth all 2-valent vertices). Therefore, each G_i is obtained by adding a single edge e to G_{i-1} such that the two endpoints of e are connected to the middle of edge(s) of G_{i-1}. It follows that G_i has exactly $3i$ edges for $i = 1, \cdots, r$. By Theorem 2.3, at least one of G_k, G_{k+1} and G_{k+2} is simplicial. $\square$

The following lemma is crucial to us (where $\gamma_{\max}(G)$ denotes the maximum genus of the graph G, and where $\gamma(R)$ denotes the genus of the imbedding corresponding to the rotation system R):

Lemma 3.2 *Let G be a 2-connected, 3-regular simplicial graph with at least 9^k edges. Then the maximum genus of G is at least $k+1$.*

PROOF. We prove the lemma by induction on the exponent k.

In Chen and Gross [ChGr 1991c], it was proved that with only five exceptions, all 2-connected graphs of maximum genus less than or equal to 1 can be characterized as "necklaces", which are all nonsimplicial graphs. Moreover, all those five exceptional graphs have fewer than 9 edges. Consequently, any 2-connected 3-regular simplicial graph G with at least 9 edges has maximum genus at least 2. Thus, the lemma is true for $k = 1$.

Now suppose that G is a 2-connected 3-regular simplicial graph with $3r$ edges, where $3r \geq 9^{k+1}$. By Lemma 3.1, there is a 2-connected, 3-regular simplicial graph G_1 such that G has an open linear homeo-synthesis from G_1 and G_1 has $3r_1$ edges, where $3r_1 = 9^k$, 9^k+3, or 9^k+6. By our inductive hypothesis, the maximum genus of the "subgraph" G_1 is at least $k+1$. If the maximum genus of G_1 is greater than $k+1$, then by Lemma 2.5, the maximum genus of G is at least $k+2$ and the inductive step goes through. Thus, without loss of generality, we suppose that the maximum genus of G_1 is $k+1$.

Let $G_1, G_2, \cdots, G_{h-1}, G_h \cong G$ be an open linear homeo-synthesis of G from G_1. Note that to obtain G_{j+1} from G_j in the synthesis, the two endpoints of the open path $G_{j+1} - G_j$ must be connected to the middle of edge(s) of G_j, since all G_{j+1}'s are 2-connected and 3-regular. Therefore, each graph G_{j+1} has exactly three more edges than the graph G_j, $j = 1, \cdots, r-1$, in the synthesis, and we have $3 \cdot (h-1) = 3r - 3r_1$. Moreover, since $3r \geq 9^{k+1}$ and $3r_1 \leq 9^k + 6$, we have $h = r - r_1 + 1 \geq 24 \cdot 9^{k-1} - 1$.

Given an imbedding of a graph, each edge e of the graph appears exactly twice on the boundaries of faces in the imbedding. Call each such an appearance a "side" of the edge e in the imbedding of the graph.

Let a be an edge-side in an imbedding, we denote by a^- the edge-side in the imbedding that is different from a and shares the same edge with a. When the two edge-sides of an edge e occur on the boundary of the same face, we will say that the two sides of the edge e are "on the same face" of the imbedding. Otherwise, we will say that the two sides of the edge e are "on different faces" in the imbedding.

Let R_1 be a rotation system corresponding to a maximum-genus imbedding of G_1 on a surface $S_{\max}$ of genus $k+1$. We construct a sequence of rotation systems $R_2, R_3, \cdots, R_{h-1}, R_h$ for the graphs $G_2, G_3, \cdots, G_{h-1}, G_h \cong G$, respectively, by adding subsequently the open paths $p_1, p_2, \cdots, p_{h-1}$, where $p_c = G_{c+1} - G_c$, for $c = 1, 2, \cdots, h-1$. We refer to the edges in the graph G_1 and the parts of them that are edges in the later graphs G_c, $c \geq 1$ as *old edges* and the edges on the open paths $p_1, p_2, \cdots, p_{h-1}$ as *new edges*.

For each number c in the range $1 \leq c \leq h-1$, let the two endpoints of the path p_c be u_c and v_c, respectively and suppose that u_c and v_c are on the middle of two edges e_{u_c} and e_{v_c} of G_c, respectively. We call these vertices u_c and v_c, $1 \leq c \leq h-1$, *new vertices*, the vertices of the graph G_1, *old vertices*. Since all graphs G_c are 3-regular, and since by our convention, 2-valent vertices are always smoothed, no old vertex can be a new vertex.

Regardless how we add the path p_c to the rotation system R_c, the genus $\gamma(R_{c+1})$ of the resulting rotation system R_{c+1} of G_{c+1} is at least as large as the genus $\gamma(R_c)$ of the rotation system R_c of G_c. Moreover, by Lemma 2.5, the maximum genus of G_{c+1} is at least as large as the maximum genus of G_c. Consequently, the genus of any rotation system R_c obtained by adding $p_1, \cdots, p_{c-1}$ to R_1, and the maximum genus of any graph G_c, for $1 \leq c \leq h$, are at least as large as the genus of the rotation system R_1 which is the maximum genus $k+1$ of the graph G_1.

If any rotation system R_c, $1 \leq c \leq h$, has genus greater than $k+1$, then the maximum genus of the graph G_c is greater than $k+1$, and thus, so is the maximum genus of the graph $G_h \cong G$, and our inductive step goes through. Therefore, without loss of generality, we assume that all rotation systems R_c, $1 \leq c \leq h$, have genus $k+1$. We now show that at least one of the rotation systems R_c, $1 \leq c \leq h$, can be modified to get a rotation system of larger genus for the corresponding graph, and thus complete our proof for the inductive step.

If a side of e_{u_c} and a side of e_{v_c} are on different faces in the rotation system R_c, then we can add the path p_c in such a way that p_c merges these two different faces in R_c. Consequently, the resulting rotation system R_{c+1} has genus

$$\gamma(R_{c+1}) = \gamma(R_c) + 1 \geq \gamma_{\max}(G_1) + 1 = k + 2$$

Therefore we have

$$\gamma_{\max}(G_h) \geq \gamma(R_h) \geq \gamma(R_{c+1}) \geq k + 2$$

and the inductive step goes through. Therefore without loss of generality, we can assume that for each c, $1 \leq c \leq h-1$, both the sides of the edges

e_{u_c} and e_{v_c} of the graph G_c are on the boundary of the same face F_c of the rotation system R_c.

According to this assumption, for each c, $1 \leq c \leq h-1$, in the rotation system R_{c+1}, the path p_c, which is a single edge in G_{c+1}, splits the face F_c of the rotation system R_c into two different faces regardless how it is added, and no path p_c would merge faces in R_c. Consequently, the two sides of p_c in R_{c+1} are in different faces, therefore in different faces in all rotation systems R_l for $l = c+1, \cdots, h$, since no path p_l would merge faces. Thus, no part of the path p_c can be used as the edges e_{u_l} and e_{v_l}, to which the endpoints of the path p_l are going to attach, in the rotation system R_l to get the rotation system R_{l+1} for $l \geq c+1$. In other words, no new vertex u_l or v_l can be located on p_c, for $l \geq c+1$. Consequently, the path p_c, $1 \leq c \leq h$, consists of a single new edge, it will be a single edge in the final graph G, and its two endpoints are located on the edge(s) of the graph G_1. Thus from now on, we can also call the paths $p_1, \cdots, p_{h-1}$ "new edges" of G. Moreover, since all edge-sides of e_{u_c} and e_{v_c} are in the same face F_c in the rotation system R_c and no new edge p_j would merge faces for $1 \leq j \leq c-1$, both the edge-sides of the edges e_{u_c} and e_{v_c} must be in the same face in the rotation system R_1. Thus the new edges p_c must be chords of a face in the rotation system R_1, for $1 \leq c \leq h-1$.

Now we have a clearer picture for the graphs G_c and the rotation systems R_c for $1 \leq c \leq h$: the rotation system R_c is obtained by adding chords $p_1, p_2, \cdots, p_{c-1}$ to the faces in the rotation system R_1. The number of edges of the graph G_1 is $3r_1$, which is at most $9^k + 6$. Now since each new edge p_c, $1 \leq c \leq h-1$, is a chord of a face in R_1 and all graphs G_c, $1 \leq c \leq h$, are 3-regular, the new edge p_c contributes exactly two new vertices to the final graph G. Therefore there are totally $2(h-1)$ new vertices created in this process when we finally reach the rotation system R_h of the graph G. Since $2(h-1) \geq 48 \cdot 9^{k-1} - 4$ which is larger than $2(9^k + 6)$ for $k \geq 1$, there must be at least one edge e of the graph G_1 on which at least three new vertices are created. Let u_x, u_y and u_z be three new vertices created on the edge e such that u_x, u_y and u_z are consecutive in this ordering on e (i.e., u_y is between u_x and u_z on e and no other new vertices are between u_x and u_z in the graph G). Suppose that u_x, u_y and u_z are the endpoints of the new edges p_x, p_y, and p_z, respectively. See Figure 1. Note that the three new edges p_x, p_y, and p_z could be chords on different sides of the edge e.

Neither the new edges p_x and p_z can be identical with the new edge p_y, otherwise, the edge p_y together with a part of the edge e will form a pair of multiple edges in the final graph G, which is simplicial. We construct a

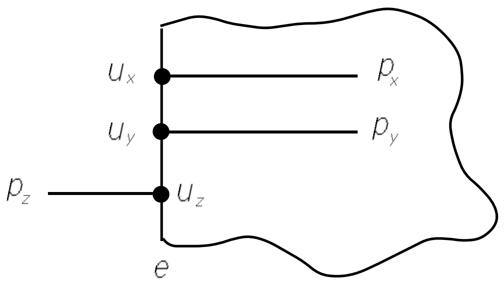

Figure 1: Edge e divided by paths p_x, p_y, and p_z

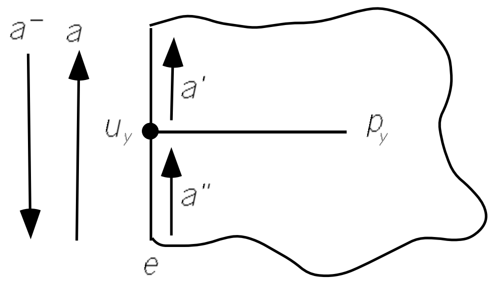

Figure 2: Edge side a divided by path p_y

rotation system $\overline{R}_h$ for $G_h \cong G$ from R_h as follows: First remove all new edges p_c, $1 \leq c \leq h-1$, from the rotation system R_h, so that we get back to the rotation system R_1 which has genus $k+1$. Now we add the new edge p_y to the rotation system R_1 and suppose we get a rotation system R' for the graph $G_1 + p_y$. If the new edge p_y can be added in such a way that p_y merges two different faces in R_1, then the resulting rotation system R' for the graph $G_1 + p_y$ has genus at least $k+2$, so the maximum genus of the final graph G is at least $k+2$. Otherwise, p_y must be a chord on a face of the rotation system R_1. Suppose in this case the new vertex u_y divides a side a of the edge e in R_1 into two edge-sides a' and a'' (without loss of generality, we suppose that exactly three new vertices were created on the edge e. If it is not the case, we can use a part of the edge instead). Since the two sides of p_y are in different faces now in the rotation system R' (see Figure 2). the edge-sides a' and a'' must be in different faces in R'. Therefore at least one of the edge-sides a' and a'' is on a face in R' which is different from the face on which the edge-side a^- lies. Without loss of generality, suppose a' and a^- are on different faces, then properly adding the new edge p_x now will merge two different faces, and get a rotation system of genus $k+2$ for the graph $G_1 + p_y + p_x$. Finally we arbitrarily add all other new edges back. The resulting rotation system $\overline{R}_h$ of the graph G will have genus at least $k+2$.

This completes the proof. □

Remark Lemma 3.2 might seem intuitively obvious at first glance. After all, the maximum genus of an increasing sequence of graphs "ought" to increase. However, additional conditions such as 2-connectedness and simplicialness are necessary. (See Gross, Klein, and Rieper [GrKlRi 1989].)

Chen, Gross, and Rieper [ChGrRi 1991] have obtained an interesting relationship between average genus and maximum genus for graphs whose vertices are of valence at most 3.

Lemma 3.3 *([ChGrRi 1991]) Let G be a graph with maximum valence at most 3. Then the average genus of G is at least half the maximum genus.*

Now we can state our main theorem for this section.

Theorem 3.4 *Let G be a 2-connected, 3-regular simplicial graph with at least 9^k edges. Then the average genus of G is at least $(k+1)/2$.*

Proof. This follows immediately from Lemmas 3.2 and 3.3. □

4 On the average genus of 2-connected simplicial graphs

A specialized form of vertex-splitting (the "inverse" of edge-contraction) within graph imbeddings facilitates the removal of the 3-regularity restriction needed for Theorem 3.4.

Let G be a simplicial graph, let v be a vertex of valence $d \geq 4$, let $e_1, \cdots,$ e_d be the edges incident on v, and let $u_1, \cdots, u_d$ be their respective other endpoints. For $i = 2, \cdots, d$, we define the graph G_i to be the supergraph obtained from $G - v$ by adjoining u_1 and u_i to a new vertex x, adjoining all the other ex-neighbors of v to a new vertex y, and adjoining x and y. Thus,

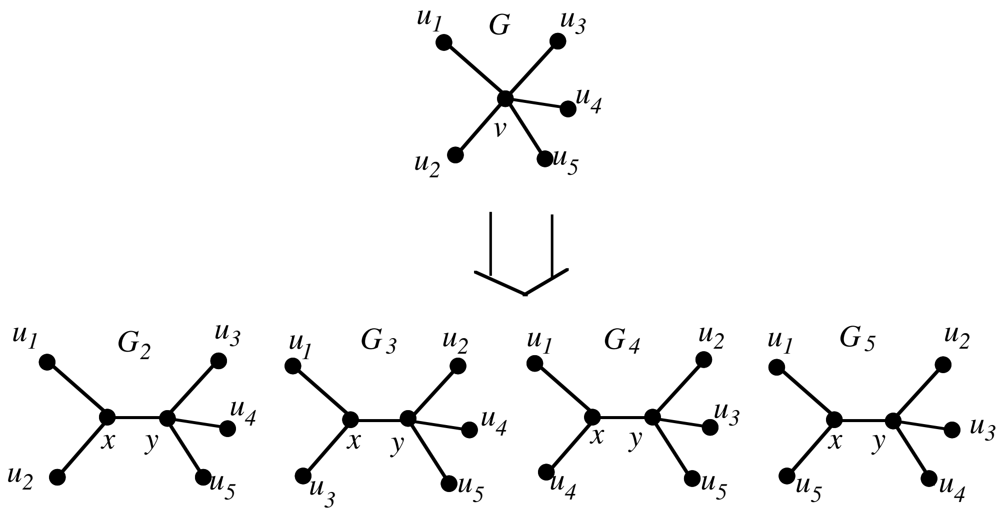

Figure 3: The proper splittings of a vertex v of valence 5 with designated neighbor u_1.

the new vertex x is 3-valent for each G_i and the new vertex y is $(d-1)$-valent, as illustrated in Figure 3. Within this section, we call G_i a "proper splitting" of G at v with "designated neighbor" u_1.

Theorem 4.1 *Any graph obtained by properly splitting a 2-connected simplicial graph at a vertex of valence at least four is also 2-connected and simplicial.*

PROOF. It is straightforward to establish that no cutpoints, multiple edges, or self-loops can be created by a proper splitting of a 2-connected simplicial graph at a vertex of valence at least four. □

To extend the operation described above to imbedded graphs, we suppose that R is a rotation system for the graph G, and that the rotation at vertex v is

$$v. \quad e_b e_1 e_a \pi$$

where π is a fixed ordering of the other edges incident on v. Then let R_a be the rotation system of graph G_a with rotations

$$x. \quad e_1 e_a e \qquad \textit{and} \qquad y. \quad e_b e \pi$$

and all other vertex rotations as in R, where e is the new edge in G_a that connects the new vertices x and y; and let R_b be the rotation system of the graph G_b with rotations

$$x.\ \ e_b e_1 e \qquad \textit{and} \qquad y.\ \ e e_a \pi$$

and all other vertex rotations as in R. We say that rotation systems R_a and R_b are obtained by "properly splitting" the vertex v in the rotation system R with the designated neighbor u_1. We also say that rotation system R is obtained by *contracting* either rotation system R_a or R_b on the edge e.

Lemma 4.2 *Let R be any rotation system for a 2-connected, simplicial graph G with a vertex v of valence $d \geq 4$ and a designated neighbor. Then there are exactly two rotation systems of the proper splittings of G at v with that designated neighbor that are proper splittings of R. Moreover, every rotation system of a proper splitting of G is uniquely contractible on the edge e to a rotation system of G.*

PROOF. Let the designated neighbor be u_1. Suppose that the rotation at v in R is

$$e_b e_1 e_a \pi$$

where π is a fixed ordering of the other edges incident on v in G. By our definition of the graphs G_i, $i = 2, \cdots, d$, R can only be obtained by contracting the edge e in the rotation systems of the graphs G_b and G_a. Moreover, if R can be obtained by contracting the edge e in a rotation system R_b of G_b, then rotations at all vertices of G_b, including the vertices x and y are determined, i.e., the rotation at x must be $e_b e_1 e$ and the rotation at y must be $e_a \pi e$ and rotations at all other vertices must be the same as in R. Moreover, if R_b is a rotation system of a graph G_b that is obtained by properly splitting the graph G at the vertex v with the designated neighbor u_1, the it is obvious that R_b can be contracted on the edge e to a unique rotation system of the graph G. $\square$

Let $\mathcal{R}_v$ be the union of sets of all rotation systems for all graphs obtained by a proper splitting of G at v with the designated neighbor u_1, and let $\mathcal{R}$ be the set of all rotation systems for the graph G. We define a map ϕ from $\mathcal{R}_v$ to $\mathcal{R}$ as follows: Given a rotation system R_i in $\mathcal{R}_v$, $\phi(R_i)$ is the rotation system of G which is obtained by contracting the edge e in R_i. By Lemma 4.2, the map ϕ is well-defined and each rotation system in $\mathcal{R}$ has exactly 2 rotation systems in $\mathcal{R}_v$ as its preimage under the map ϕ. Moreover, it is easy to see that a rotation system R in $\mathcal{R}_v$ and the corresponding image $\phi(R)$ in $\mathcal{R}$ have the same genus. Finally, if we suppose that the number of rotation systems of G is $|\mathcal{R}| = \delta(d-1)!$, where δ is an integer, then the number of rotation systems of each G_i, $i = 2, \cdots, d$, is $2\delta(d-2)!$. Therefore, the total number of rotation systems in the set $\mathcal{R}_v$ is $|\mathcal{R}_v| = 2\delta(d-1)! = 2|\mathcal{R}|$.

Given a graph G, we say that the *maximum valence* of G is d if the maximum valence of vertices in G is d.

With these preparations, we now are able to prove the following important theorem, a generalization of Theorem 3.4.

Theorem 4.3 *Let G be a 2-connected simplicial graph with at least 9^k edges. Then $\gamma_{avg}(G) \geq (k+1)/2$.*

PROOF. Suppose that the maximum valence of the graph G is at most d. We prove the theorem by induction on d. The underlying idea is that the average genus of G equals the weighted mean of the values of average genus of the various splittings.

First note that a 2-connected graph cannot contain vertices of valence 1 and by our convention, no vertices in graphs can have valence 2. In other words, each vertex of G has valence at least 3. Therefore, we can suppose that $d \geq 3$.

If $d = 3$, then G is a 2-connected, 3-regular and simplicial graph. By Theorem 3.4, the theorem is true.

We prove the theorem for the case $d \geq 4$ by induction on the number p of vertices in G that have valence exactly d (we are using double induction here). If $p = 0$, then the maximum valence of G is actually at most $d-1$. By the hypothesis of the first induction, the average genus of G is at least $(k+1)/2$. Now suppose $p \geq 1$. Let v be a vertex of G of valence d. We construct the $d-1$ graphs $G_2, G_3, \cdots, G_d$ by properly splitting the vertex v with a designated neighbor. By Theorem 4.1, each G_i is 2-connected and simplicial. It is also clear that each G_i contains at least $9^k + 1$ edges, the maximum valence of each G_i is at most d, and each G_i has $p-1$ vertices of valence d. By the hypothesis of our inner induction, all these graphs G_i have average genus at least $(k+1)/2$.

Now by Theorem 4.2, each rotation system R in the set $\mathcal{R}$, which is the collection of all rotation systems for the graph G, corresponds to exactly two rotation systems R_a and R_b in the set $\mathcal{R}_v$, which is the collection of all rotation systems for all graphs $G_2, G_3, \cdots, G_d$, and $\gamma(R_a) = \gamma(R_b) = \gamma(R)$.

Therefore we have (where $\gamma(R)$ is the genus of the rotation system R, $|\mathcal{S}|$ is the number of elements in the set $\mathcal{S}$, and $\mathcal{R}_i$ is the set of all rotation systems for the graph G_i):

$$\begin{aligned}
\gamma_{avg}(G) &= \frac{\sum_{R\in\mathcal{R}} \gamma(R)}{|\mathcal{R}|} && (1)\\
&= \frac{2\sum_{R\in\mathcal{R}} \gamma(R)}{2|\mathcal{R}|} && (2)\\
&= \frac{\sum_{R\in\mathcal{R}_v} \gamma(R)}{|\mathcal{R}_v|} && (3)\\
&= \frac{\sum_{i=2}^{d}\sum_{R\in\mathcal{R}_i} \gamma(R)}{\sum_{i=2}^{d}|\mathcal{R}_i|} && (4)\\
&= \frac{1}{d-1}\sum_{i=2}^{d}\frac{\sum_{R\in\mathcal{R}_i} \gamma(R)}{|\mathcal{R}_i|} && (5)\\
&= \frac{1}{d-1}\sum_{i=2}^{d}\gamma_{avg}(G_i) && (6)\\
&\geq \frac{1}{d-1}\sum_{i=2}^{d}\frac{k+1}{2} && (7)\\
&= \frac{k+1}{2} && (8)
\end{aligned}$$

To obtain (5) from (4), we have used the fact that $|\mathcal{R}_i|$ is independent of i. □

There are several important corollaries to Theorem 4.3.

Theorem 4.4 *For every integer k, only finitely many 2-connected simplicial graphs have average genus less than k.*

Proof. By Theorem 4.3, a 2-connected simplicial graph cannot have average genus less than k unless it has fewer than 9^{2k} edges. However, the number of 2-connected simplicial graphs with at most 9^{2k} edges is clearly finite. □

Corollary 4.5 *Let r be any real number. Then only finitely many real numbers less than r are possible values of average genus for a 2-connected simplicial graph.*

Corollary 4.6 *Let r be any real number. Then only finitely many 2-connected simplicial graphs have average genus equal to r.*

Corollary 4.7 *The set of possible values of average genus for the 2-connected simplicial graphs has no limit points.*

5 On the average genus of 3-connected graphs

For the class of 3-connected graphs, we can eliminate the restriction of simplicialness, and still obtain results like those of Section 4.

We start with 3-connected, 3-regular graphs. A 3-connected 3-regular graph G cannot contain any self-loops because deleting an endpoint of an edge whose other endpoint is holding a self-loop would disconnect the graph. Moreover, a 3-connected 3-regular graph G cannot contain multiple edges: suppose otherwise e_1 and e_2 are a pair of multiple edges connecting the vertices v_1 and v_2 of G. Let $\{v_1, v_3\}$ and $\{v_2, v_4\}$ be the other two edges of G that are incident on v_1 and v_2, respectively. Then deleting the vertices v_3 and v_4 from G would disconnect the graph. Therefore, a 3-connected, 3-regular graph must be simplicial. Therefore, by Theorem 3.4, we immediately infer the following result.

Theorem 5.1 *Let G be a 3-connected, 3-regular graph with at least 9^k edges. Then the average genus of G is at least $(k+1)/2$.*

For a similar result for general 3-connected graphs, the two essential forms of non-simplicialness are a preliminary consideration. We recall that a graph with two vertices and n edges adjoining them is called a *dipole* and is denoted D_n. Also, a graph with one vertex and n self-loops is called a *bouquet* and is denoted B_n.

Definition The *Stirling number $s(n,k)$ of the first kind* (or *Stirling cycle number*) is defined to be the number of permutations that have exactly k cycles in the symmetric group Σ_n of order n.

We first need a few basic formulas for Stirling cycle numbers.

Lemma 5.2

$$\sum_{k=0}^{n} s(n,k) = n! \tag{9}$$

$$\sum_{k=0}^{n}(-1)^k s(n,k) = 0 \tag{10}$$

$$s(n,0) = 0 \tag{11}$$

$$s(n,1) = (n-1)! \tag{12}$$

$$s(n,2) = H_{n-1}\cdot(n-1)! \tag{13}$$

$$s(n+1,m+1) = \sum_{k=0}^{n}\binom{k}{m}s(n,k) \tag{14}$$

$$(-1)^m s(n,m) = \sum_{k=0}^{n}(-1)^k\binom{k}{m}s(n+1,k+1) \tag{15}$$

$$\sum_{k\ even} s(n,k) = \sum_{k\ odd} s(n,k) = \frac{n!}{2} \tag{16}$$

By H_n we mean the Harmonic number: $H_n = \sum_{k=1}^{n}\frac{1}{k}$. We have a relationship between H_n and $\ln n$:

$$\ln n < H_n < \ln n + 1$$

PROOF. The proofs for all these equalities except (16) can be found in [GrKnPa 1989]. The equality (16) follows directly from equality (9) and equality (10). □

Lemma 5.3

$$\sum_{k=0}^{n} k\cdot s(n,k) = H_n\cdot n! \tag{17}$$

$$\sum_{k=0}^{n}(-1)^k k\cdot s(n,k) = (n-2)! \tag{18}$$

PROOF. From equality (14) in Lemma 5.2, let $m=1$, then apply the equality (13) to obtain the equality (17). From equality (15), let $m=1$, then apply equality (12) and equality (10), to obtain the equality (18). □

Lemma 5.4

$$\sum_{k\ even} k\cdot s(n,k) = \frac{1}{2}H_n n! + \frac{1}{2}(n-2)! \tag{19}$$

$$\sum_{k\ odd} k \cdot s(n,k) = \frac{1}{2} H_n n! - \frac{1}{2}(n-2)! \tag{20}$$

PROOF. From Lemma 5.3 directly. $\square$

Lemma 5.5 *For every sufficiently large number n, $\gamma_{avg}(D_n) \geq n/4$.*

PROOF. Since $\ln n < H_n < \ln n + 1$, there must be an integer Q_1 such that when $n > Q_1$ we have

$$\frac{n}{2} > H_{n+1} + \frac{1}{n(n+1)}$$

Let $g_r(D_n)$ be the number of imbeddings of D_n that have exactly r faces, i.e., that have genus $\frac{n-r}{2}$. Rieper has obtained [Rieper 1987]

$$g_r(D_n) = \begin{cases} \frac{2(n-1)!}{n(n+1)} s(n+1,r) & \text{if } n \equiv r \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore when n is even, the average genus of D_n is

$$\begin{aligned} \gamma_{avg}(D_n) &= \frac{1}{((n-1)!)^2} \sum_{r\ even} \frac{n-r}{2} \cdot \frac{2(n-1)!}{n(n+1)} s(n+1,r) \\ &= \frac{1}{((n-1)!)^2} \left\{ \frac{(n-1)!}{n+1} \sum_{r\ even} s(n+1,r) - \right. \\ &\qquad \left. \frac{(n-1)!}{n(n+1)} \sum_{r\ even} r \cdot s(n+1,r) \right\} \\ &= \frac{n}{2} - \frac{H_{n+1}}{2} - \frac{1}{2n(n+1)} \end{aligned}$$

where the equality (16) of Lemma 5.2 and equality (19) of Lemma 5.4 are used. Now when $n > Q_1$, we have $\gamma_{avg}(D_n) > n/4$.

Similarly, when n is odd, the average genus of D_n is

$$\gamma_{avg}(D_n) = \frac{n}{2} - \frac{H_{n+1}}{2} + \frac{1}{2n(n+1)}$$

In any case, we always have $\gamma_{avg}(D_n) > n/4$ when $n > Q_1$. $\square$

Lemma 5.6 *For every sufficiently large number n, $\gamma_{avg}(B_n) \geq n/4$.*

PROOF. Let $b(n,r)$ be the number of imbeddings of B_n that have exactly

r faces, i.e., that have genus $(1+n-r)/2$. Stahl [Stahl 1989] has proved that

$$\lim_{n\to\infty}\left\{\frac{1}{(2n-1)!}\sum_{r=1}^{n+1}(r\cdot b(n,r)) - H_{2n}\right\} = 0$$

We let Q_2 be a large integer such that when $n \geq Q_2$ we have $\frac{n}{4} > H_{2n}$ and

$$\sum_{r=1}^{n+1} r\cdot b(n,r) - H_{2n}\cdot(2n-1)! < \frac{1}{2}(2n-1)!$$

Then for $n \geq Q_2$, the average genus of B_n is (note that $\sum_{r=1}^{n+1} b(n,r) = (2n-1)!$)

$$\begin{aligned}
\gamma_{avg}(B_n) &= \frac{1}{(2n-1)!}\sum_{r=1}^{n+1}\frac{1+n-r}{2}b(n,r) \\
&= \frac{1}{(2n-1)!}\left\{\frac{n+1}{2}\sum_{r=1}^{n+1}b(n,r) - \sum_{r=1}^{n+1}r\cdot b(n,r)\right\} \\
&\geq \frac{1}{(2n-1)!}\left\{\frac{n+1}{2}(2n+1)! \right. \\
&\qquad \left. -H_{2n}(2n-1)! - \frac{1}{2}(2n-1)!\right\} \\
&= \frac{n}{2} - H_{2n} \\
&> \frac{n}{4}
\end{aligned}$$

□

Now we are ready for the following theorem.

Theorem 5.7 *For a sufficiently large number k, every 3-connected graph G with at least $9^{2k}(k+1)^2$ edges has average genus at least $(k+1)/2$.*

Proof. We shall prove that the graph G contains either a 3-connected simplicial subgraph with a large number of edges or, alternatively, a large bouquet or a large dipole.

Let $Q = \max\{Q_1, Q_2\}$, where Q_1 and Q_2 are the integers in the proofs of Lemma 5.5 and Lemma 5.6, respectively. Therefore, when $n \geq Q$, both the average genus of D_n and the average genus of B_n are larger than $n/4$. Now suppose $k \geq Q$, and also, without loss of generality, suppose that $Q \geq 1$.

If the graph G is simplicial, then the theorem is implied by Theorem 4.3. Accordingly, we suppose that G contains some self-loops or multiple edges. There are two different cases.

(1) The graph G contains at least $6 \cdot 9^{k-1}$ vertices.

Then we delete all self-loops, and for each multi-edge adjacency, we delete all but one edge. Let the resulting graph be G_1. Note that such operations do not destroy the 3-connectivity of the graph. Consequently, each vertex of G has valence at least 3 in the new graph G_1 and no vertex of G could be smoothed in G_1. Therefore, the graph G_1 is 3-connected, simplicial, and contains at least $6 \cdot 9^{k-1}$ vertices. Since each vertex of G_1 has valence at least 3, the number of edges in G_1 is at least 9^k. By Theorem 4.3 again, the average genus of G_1 is at least $(k+1)/2$. Now the theorem follows by Lemma 2.4 since G_1 is a subgraph of G.

(2) The graph G contains fewer than $6 \cdot 9^{k-1}$ vertices.

In this case, we have two subcases. If G contains at least $12 \cdot 9^{k-1}(k+1)$ self-loops, then there is at least one vertex v of G such that there are more than $2(k+1)$ self-loops located on v. Consequently, G contains a subgraph which is a bouquet of $2k+3$ circles which has the average genus greater than $(k+1)/2$ by Lemma 5.6. Therefore the average genus of G is also greater than $(k+1)/2$.

Finally, suppose that G has less than $6 \cdot 9^{k-1}$ vertices and less than $12 \cdot 9^{k-1}(k+1)$ self-loops. Then there are at least

$$\begin{aligned} F(k) &= 9^{2k}(k+1)^2 - 12 \cdot 9^{k-1}(k+1) \\ &= 9^{k-1}(k+1)(9^{k+1}(k+1) - 12) \end{aligned}$$

edges of G that are not self-loops. There are at most

$$E(k) = \binom{6 \cdot 9^{k-1}}{2} = 3 \cdot 9^{k-1}(6 \cdot 9^{k-1} - 1)$$

different pairs of vertices in G. Therefore, at least one pair of vertices of G is linked by

$$F(k)/E(k) = \frac{(k+1)(9^{k+1}(k+1) - 12)}{3(6 \cdot 9^{k-1} - 1)} > 2(k+1)$$

edges, i.e., the graph G contains a subgraph that is a dipole of $2k+3$ edges which has average genus at least $(k+1)/2$ by Lemma 5.5. $\square$

It follows that the theorems we have derived in Section 4 for 2-connected simplicial graphs are also true for 3-connected graphs.

Theorem 5.8 *For each integer k, only finitely many 3-connected graphs have average genus less than k.*

Corollary 5.9 *Let r be any real number. Then only finitely many real numbers less than r are values of average genus for any 3-connected graph.*

Corollary 5.10 *Let r be any real number. Then only finitely many 3-connected graphs have average genus equal to r.*

Corollary 5.11 *The set of possible values of average genus for the 3-connected graphs has no real limit points.*

6 Conclusions and directions for further research

Obviously, the results of the previous two sections cannot be true for arbitrarily connected graphs, since there are infinitely many different trees, all with average genus zero. Moreover, there are infinitely many 2-connected graphs of average genus less than 1, and infinitely many different possible values of average genus that are less than 1. In fact, the number 1 is an upper limit point of values of average genus for 2-connected graphs, as demonstrated by Gross, Klein, and Rieper [GrKlRi 1989]. In a sequel, we will present a systematical method, similar to Klein's construction of necklaces, to construct limit points for values of average genus of 2-connected graphs. We are able to prove that the necklace construction is essentially the only way to manufacture arbitrary large 2-connected graphs with average genus below a fixed bound and to make limit points for values of average genus of 2-connected graphs. See Chen and Gross [ChGr 1991b].

Although we have only discussed a topological invariant of graphs for graph homeomorphism type, it is not hard to see that an efficiently computable topological invariant also implies an efficient algorithm for graph isomorphism testing, as it is easy to prove that there is a polynomial-time algorithm for graph homeomorphism testing if and only if there is a polynomial-time algorithm for graph isomorphism testing. Therefore, it will lost no generality to concentrate on graph homeomorphism testing problem. We rank graph invariants for graph homeomorphism testing by two criteria: their computation time and their capacity to distinguish non-homeomorphic graphs. As we have mentioned in Section 1, no known complete invariant is known to be computable in polynomial time.

The average genus of graphs, or the combination of average genus of graphs with some other graph invariants, could be a good candidate for a complete invariant, at least for 2-connected simplicial graphs and for 3-connected graphs. Given a graph G, the average genus of G can be estimated by sampling, and sampling can be done efficiently using a "probabilistic" machine model on which random numbers can be generated. However, if the difference of possible values of average genus could be arbitrarily small, we would not be able to make decisions based on the estimated values. Fortunately, our results in this paper tells us this is not the case, at least for 2-connected simplicial graphs and for 3-connected graphs, because there are only finitely many real numbers in each finite real interval that are values of average genus, and there are no limit points for values of average genus of those classes of graphs. It would be interesting to know the minimum gap between the numbers within a finite real interval that are realizable values of average genus for 2-connected simplicial graphs and for 3-connected graphs.

Our results also show that for either the class of 2-connected simplicial graphs or the class of 3-connected graphs, there are only finitely many non-homeomorphic graphs with any fixed real number as their common average genus. Therefore, with a negligible chance, two non-homeomorphic graphs can have the same value of average genus. It would be quite interesting to derive a tight upper bound on the number of such graphs that share a value of average genus, and to characterize the graphs sharing the same value of average genus.

Such improvements might lead directly to a probabilistic solution to the graph isomorphism problem. Moreover, we are surely not constrained to using invariants as low in the hierarchy of Gross and Furst [GrFu 1987] as average genus. We might not need to rise to invariants as large as the stratified graphs of Gross and Tucker [GrTu 1989] to find a complete invariant. Moreover, we might not need to rise all the way to a complete invariant to do definitive isomorphism testing.

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