Kuratowski-Type Theorems for Average Genus

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Abstract

Graphs of small average genus are characterized. In particular, a Kuratowski-type theorem is obtained: except for finitely many graphs, a cutedge-free graph has average genus less than or equal to 1 if and only if it is a necklace. We provide a complete list of those exceptions. A Kuratowski-type theorem for graphs of maximum genus 1 is also given. Some of the methods used in obtaining these results involve variations of a classical result of Whitney.

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1 Introduction

By the *average genus* of a graph G, we mean the average value of the genus of the imbedding surface, taken over all orientable imbeddings of G. This value is evidently a rational number, and it is clearly an invariant of the homeomorphism type of a graph.

The average genus of individual graphs is in the Gross-Furst hierarchy [GrFu 1987] and was studied by Gross, Klein, and Rieper [GrKlRi 1989] (Stahl [Stahl 1983] studied the related problem of the average genus of a class of graphs with the same number of edges). [GrKlRi 1989] proved that the smallest positive values of average genus are

$$1/3,\ 1/2,\ 5/9,\ 2/3,\ 19/27,\ 3/4$$

and that there are infinitely many real numbers less than 1 that are values of average genus of 2-connected graphs. At the end of that paper, it was asked that what kind of graphs can have average genus less than 1.

In the present paper, we investigate the properties of graphs with average genus less than or equal to 1. Using a classical result of Whitney [Whitney 1932], we are able to prove that with only finitely many exceptions, a cutedge-free graph of average genus less than 1 must be a necklace. We also list all those exceptional graphs. This result gives us several interesting corollaries. For example, we are able to give a formula for all real numbers which are less than 1 and values of average genus. We are also able to prove that for any real number $\epsilon > 0$, there are only finitely many real numbers in the interval $[0, 1-\epsilon]$ that are values of average genus. Finally, we also give a Kuratowski-type theorem for graphs of maximum genus 1, which is analogous to a theorem derived by Nordhaus, Ringeisen, Stewart and White [NRSW 1972].

The outline of this paper is as follows. Section 2 introduces the basic results of theory of imbedding distribution of a graph, the concept of linear synthesis, and a classical result of Whitney. Section 3 discusses 2-connected graphs with average genus less than or equal to 1. Section 4 extends the results of Section 3 to cactus-free graphs. Section 5 gives a Kuratowski-type theorem for graphs of maximum genus 1. Some concluding remarks are given in Section 6.

2 Preliminaries

It is assumed that the reader is somewhat familiar with topological graph theory. We briefly review the fundamentals. For further description, see Gross and Tucker [GrTu 1987], or White [White 1984].

A *graph* may have multiple adjacencies or self-adjacencies. An *imbedding* must have the "cellularity property" that the interior of every region is simply connected. The closed orientable surface of genus n is denoted S_n.

A *rotation* at a vertex v is a cyclic permutation of the edge-ends incident on v. Thus, a d-valent vertex admits $(d-1)!$ rotations. A list of rotations, one for each vertex of the graph, is called a *rotation system.*

An imbedding of a graph G in an orientable surface induces a rotation system, as follows: the rotation at vertex v is the cyclic permutation corresponding to the order in which the edge-ends are traversed in a orientation-preserving tour around v. Conversely, by the Heffter-Edmonds principle, every rotation system induces a unique imbedding of G into an orientable surface. The bijectivity of this correspondence implies that the number of different ways to imbed a graph of valence sequence $d_1, \cdots, d_n$ into a closed, orientable surface is

$$\prod_{i=1}^{n}(d_i-1)!$$

For any graph G, if the number of imbeddings in the surface S_k is denoted g_k, then the sequence

$$g_0, g_1, g_2, \cdots$$

is called the *genus distribution* of G. Genus distributions for special families of graphs have been calculated by Furst, Gross, and Statman [FuGrSt 1989] and by Gross, Robbins, and Tucker [GrRoTu 1989], by blending topology and combinatorial enumeration.

The *average genus* of G is defined to be the value

$$\gamma_{avg}(G) = \frac{\sum_{i=0} i \cdot g_i}{\sum_{i=0} g_i}$$

Now we introduce the concept of linear synthesis of graphs, which originates from a classical work by Whitney [Whitney 1932], and we review Whitney's theorem that an undirected graph is 2-connected if and only if it has an open linear synthesis.

Definition Let $G = (V, E)$ be a graph. A *linear synthesis* $L = [P_0, \cdots, P_r]$

of G from P_0 is a partition of E into an ordered collection of edge disjoint subgraphs $P_0, \cdots, P_r$ of G, such that P_0 is a subgraph, each P_i is a simple path for $i = 1, \cdots, r$, and each end point of P_i, $i = 1, \cdots, r$, is contained in some P_j, with $j < i$. The P_i's, $i = 1, \cdots, r$, are called the *paths* of L and r the *length* of L. We call L an *open linear synthesis* if all paths P_i, $i = 1, \cdots, r$, are open (i.e., not a simple cycle).

It should be noticed that adding a path to a graph increases the cycle rank of the graph by 1. Therefore, the length of a linear synthesis of a graph from a simple cycle equals the dimension of its cycle subspace minus 1; thus, it is a topological invariant under graph homeomorphism.

Whitney established the following interesting relationship between graph connectivity and linear synthesis of graphs.

Theorem 2.1 *([Whitney 1932])*[3] *A graph G without self-loops has an open linear synthesis from a simple cycle if and only if G is 2-connected.*

We shall use the following two variations of Whitney's theorem (see [Chen 1990]):

Theorem 2.2 *A graph G without self-loops is 2-connected if and only if G has an open linear synthesis from every 2-connected subgraph.*

Theorem 2.3 *A graph G has a linear synthesis from a simple cycle if and only if G is cutedge-free.*

The following lemma is also crucial to our proofs.

Lemma 2.4 *The average genus of a graph is not less than the average genus of any of its subgraphs.*

Proof. Let G_1 be a subgraph of a graph G. Without loss of generality, we suppose that G is obtained by adding an edge e to G_1. Each rotation system R_1 of G_1 can be extended to c rotation systems of G, where c is a

[3] In his seminal work [Whitney 1932], Whitney neither mentioned this result nor introduced the terminology "linear synthesis" explicitly. What he proved is that any 2-connected graph of cycle rank greater than 1 contains a path whose removal results in a 2-connected graph. Linear synthesis of graphs has recently found applications in theoretical computer science where the terminology "ear decomposition" is used instead of "linear synthesis" [MiRa 1987].

fixed constant (if the two endpoints of e are connected to two vertices v_1 and v_2 of G_1, of valence d_1 and d_2, respectively, then $c = d_1 \cdot d_2$ if v_1 and v_2 are different, and $c = d_1(d_1+1)$ if v_1 and v_2 are identical). Each such rotation system of G has genus at least as large as the genus of rotation system R_1 of G_1. Therefore, the average genus of G is at least as large as

$$\frac{c \cdot \gamma_{avg}(G_1) \cdot N_{G_1}}{c \cdot N_{G_1}} = \gamma_{avg}(G_1)$$

where N_{G_1} is the number of rotation systems of G_1. $\square$

Necklaces are a class of graphs developed by Klein (see [GrKlRi 1989]). We give the formal definition here.

Definition A *necklace of type* $(r, 0)$ is a graph $N_{(r,0)} = (V, E)$ with vertex set $V = \{v_1, v_2, \cdots, v_{2r}\}$, such that the vertex v_{2s-1} is connected by a single edge to the vertex v_{2s}, for $s = 1, \cdots, r$, and that the vertex v_{2s-2} is connected by two multiple edges to the vertex v_{2s-1}, where v_0 is the same as v_{2r}. A *necklace of type* (r, s) is the necklace $N_{(r,0)}$ of type $(r, 0)$ plus s self-loops added at s distinct interior points of those nonmultiple edges of $N_{(r,0)}$ (in an arbitrary way).

The graphs in Figure 1 are two necklaces of type $(4, 3)$. Note that there is a unique necklace of type $(r, 0)$ up to homeomorphism. However, there are, in general, more than one non-homeomorphic necklace of type (r, s) for $s > 1$.

3 2-connected graphs with small average genus

Gross, Klein and Rieper [GrKlRi 1989] have studied 3-connected graphs with small average genus. They proved that K_4 is the only 3-connected graph of average genus less than 1. They also showed, however, that there are infinitely many 2-connected graphs of average genus less than 1. In this section, after an extension of the results of Gross, Klein and Rieper [GrKlRi 1989], we characterize all 2-connected graphs of average genus less than 1.

To study 2-connected graphs with small average genus, we apply Whitney's theorem, starting with a simple cycle. We consider all possible ways to subsequently add paths, and we examine the average genus of all resulting

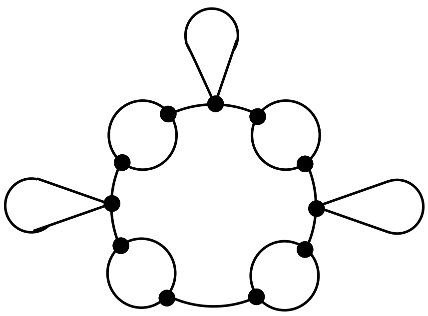

Figure 1: Two necklaces of type $(4, 3)$

graphs. Since Whitney's theorem is for loopless graphs, we first delete all self-loops. Later we shall see how adding loops increases the average genus. The first lemma is straightforward:

Lemma 3.1 *Removing all the self-loops from a 2-connected graph yields a 2-connected graph.*

PROOF. A graph G is 2-connected if and only if there is no cutpoint in G. However, it is straightforward to establish that deleting a self-loop in a graph does not create a cutpoint. $\square$

A necklace is obviously 2-connected. The following lemma of Gross, Klein, and Rieper [GrKlRi 1989] gives the average genus of necklaces.

Lemma 3.2 *The average genus of a necklace $N_{(r,s)}$ of type (r, s) equals $1 - \frac{2^{s-r}}{3^s}$.*

PROOF. Since $N_{(r,s)}$ has $2r$ 3-valent vertices and s 4-valent vertices, it follows that the total number of orientable imbeddings (i.e. the total number of rotation systems) is $4^r 6^s$. It is also easy to see that a spanning tree of $N_{(r,s)}$ must be a Hamiltonian path in $N_{(r,s)}$. Therefore, the cotree has $r+s-1$ odd components. Now it is easily derived from Xuong's theorem [Xuong 1979] that the maximum genus of $N_{(r,s)}$ is 1. Accordingly, our remaining task is to determine the number of planar imbeddings of $N_{(r,s)}$.

In order to count the planar imbeddings of $N_{(r,s)}$, let us suppose that C is a maximum simple cycle in $N_{(r,s)}$. Thus C is a Hamiltonian cycle that contains one edge of each doubled edge-pair and none of the self-loops. In the plane, we draw a rotation projection (see [GrTu 1987]) for $N_{(r,s)}$ so that C lies on a circle. The corresponding imbedding is planar if and only if both ends of each of the r other edges of a doubled-edge pair lie on the same side of C and both ends of each self-loop lie on the same side of C. It follows that the number of planar imbeddings is 2^r4^s.

Therefore, the average genus of $N_{(r,s)}$ is

$$\frac{4^r6^s - 2^r4^s}{4^r6^s} = 1 - \frac{2^{s-r}}{3^s}$$

□

Corollary 3.3 *There are infinitely many 2-connected graphs with average genus less than 1. In particular, all necklaces have average genus less than 1.*

We introduce special notations for some interesting graphs. A *bouquet* B_n of n edges is the graph with one vertex and n self-loops. A *dipole* of n edges is the graph with two vertices and n adjacencies between them.

Now let us apply Whitney's theorem to all 2-connected graphs of small average genus. We first suppose that the graphs we are considering contain no self-loops. Such a 2-connected graph G has an open linear synthesis

$$P_0, P_1, \cdots, P_r$$

from the simple cycle P_0. We will construct G by drawing the paths $P_0, P_1, \cdots, P_r$. Let the notation $P_0 + P_1 + \cdots + P_i$ denote the smallest subgraph of G containing all edges in the paths $P_0, P_1, \cdots, P_i$.

Definition Class $2C_r$ contains all 2-connected graphs obtainable from a cycle by an open linear synthesis of length r.

The average genus of each graph considered below is computed explicitly. In fact, a computer program helped us to confirm our calculations.

If $r = 0$, then G is homeomorphic to a simple cycle, which has average genus 0. If $r = 1$, then G is homeomorphic to a dipole D_3 of 3 edges, which has average genus $\frac{1}{2}$.

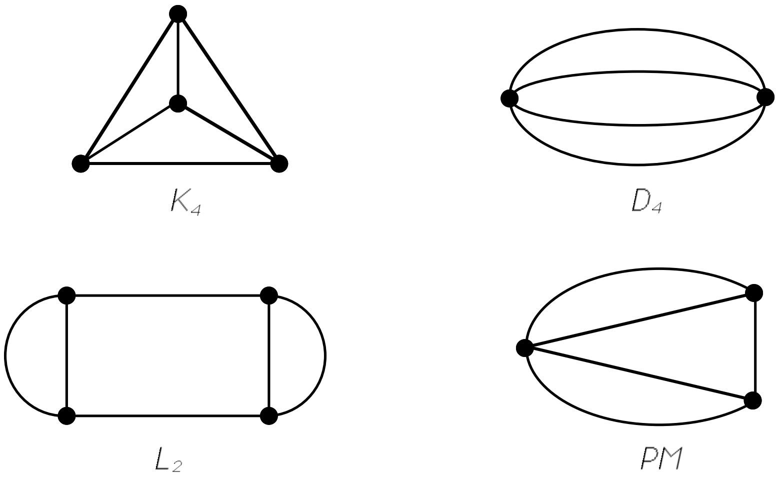

Figure 2: The four 2-connected graphs in class $2C_2$

Remark The dipole D_3 is the only 2-connected graph in class $2C_1$.

Now there are four non-homeomorphic graphs that can be obtained by adding an open path e to the dipole D_3: the two ends of e are connected to the middles of two different edges of D_3, the two ends of e are connected to the two vertices of D_3, the two ends of e are connected to the middle of a single edge of D_3, or one end of e is connected to the middle of one edge of D_3 and the other end of e is connected to one vertex of D_3. Figure 2 illustrates these four resulting graphs. These four graphs are K_4, D_4, L_2, and PM.[4]

The first graph is the complete graph K_4, which has average genus $\frac{7}{8}$. There are six non-homeomorphic graphs that can be obtained by adding an open path e to K_4: if one end of e is connected to a vertex v of K_4, then the other end of e could be: (1) connected to the middle of an edge that is incident on v, (2) connected to the middle of an edge that is not incident on v, or (3) connected to another vertex of K_4. If neither end of e is connected

[4]All these names are standard except the last one: We call it PM because it looks like a $PacMan$.

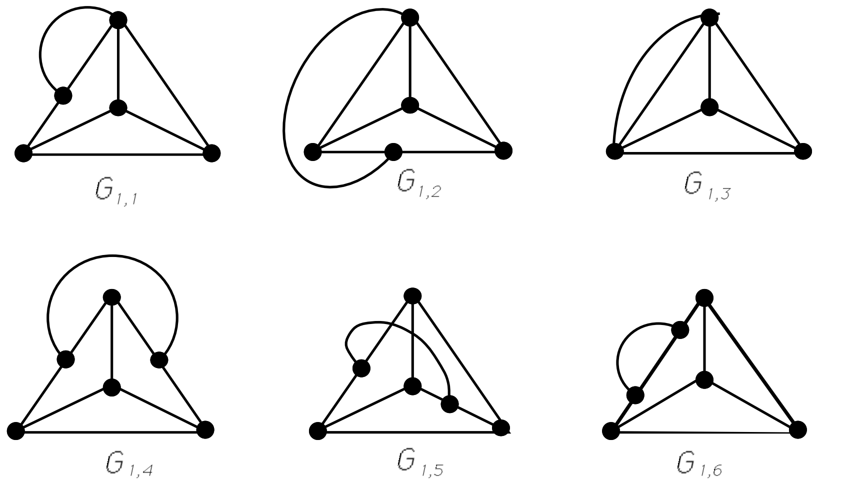

Figure 3: The six open linear extensions of K_4
(all $\gamma_{avg} > 13/12$)

to a vertex of K_4, then either (4) the two ends of e are connected to the middles of two adjacent edges of K_4, or (5) the two ends of e are connected to the middles of two nonadjacent edges of K_4, or (6) the two ends of e are connected to the middle of a single edge of K_4. Figure 3 lists all these six graphs. Call these graphs $G_{1,1}$, $G_{1,2}$, $G_{1,3}$, $G_{1,4}$, $G_{1,5}$, and $G_{1,6}$, respectively.

It is routine to compute the average genus of these graphs. We get

$$\gamma_{avg}(G_{1,1}) = \frac{31}{24} \qquad \gamma_{avg}(G_{1,2}) = \frac{65}{48}$$

$$\gamma_{avg}(G_{1,3}) = \frac{4}{3} \qquad \gamma_{avg}(G_{1,4}) = \frac{43}{32}$$

$$\gamma_{avg}(G_{1,5}) = \frac{11}{8} \qquad \gamma_{avg}(G_{1,6}) = \frac{19}{16}$$

Now we consider the dipole D_4 of four edges that has average genus $\frac{5}{6}$. There are four non-homeomorphic graphs that can be obtained by adding an open path e to D_4: if one end of e is connected to a vertex of D_4, then the other end of e can be connected either to the middle of one edge of D_4 or to the other vertex; if both ends of e are not connected to a vertex of D_4,

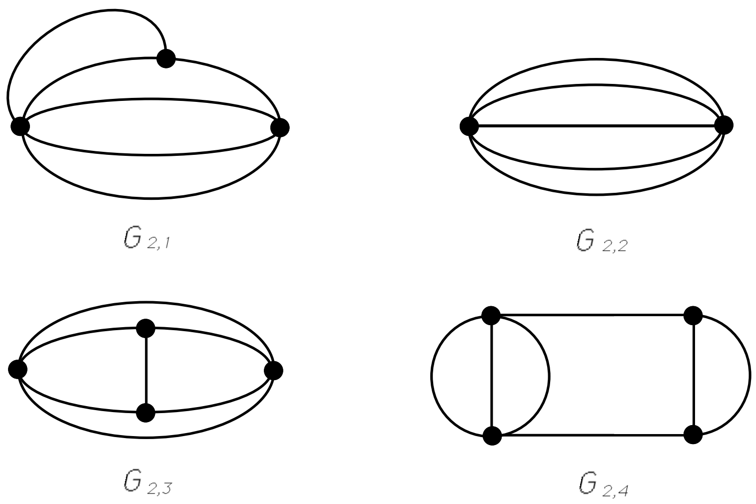

Figure 4: The four open linear extensions of D_4
(all $\gamma_{avg} > 13/12$)

then they can be either on the middles of two different edges of D_4 or on the middle of a single edge of D_4. Figure 4 lists all these four graphs. Call these graphs $G_{2,1}$, $G_{2,2}$, $G_{2,3}$, and $G_{2,4}$, respectively.

We also compute the average genus for these graphs.

$$\gamma_{avg}(G_{2,1}) = \frac{31}{24} \qquad \gamma_{avg}(G_{2,2}) = \frac{31}{24}$$

$$\gamma_{avg}(G_{2,4}) = \frac{7}{6}$$

Note also that $G_{2,3}$ is homeomorphic to $G_{1,3}$ which has average genus $\frac{4}{3}$.

Careful itemization of extensions and combination of homeomorphic cases show that there are ten non-homeomorphic graphs that can be obtained by adding an open path to the graph L_2 (L_2 has average genus $\frac{3}{4}$). Figure 5 depicts all these graphs. Call these graphs $G_{3,1}$, $G_{3,2}$, $\cdots$, $G_{3,10}$, respectively.

The graph $G_{3,2}$ is homeomorphic to the graph $G_{2,4}$ which has average genus $\frac{7}{6}$. The graph $G_{3,6}$ is homeomorphic to $G_{1,1}$ which has average genus

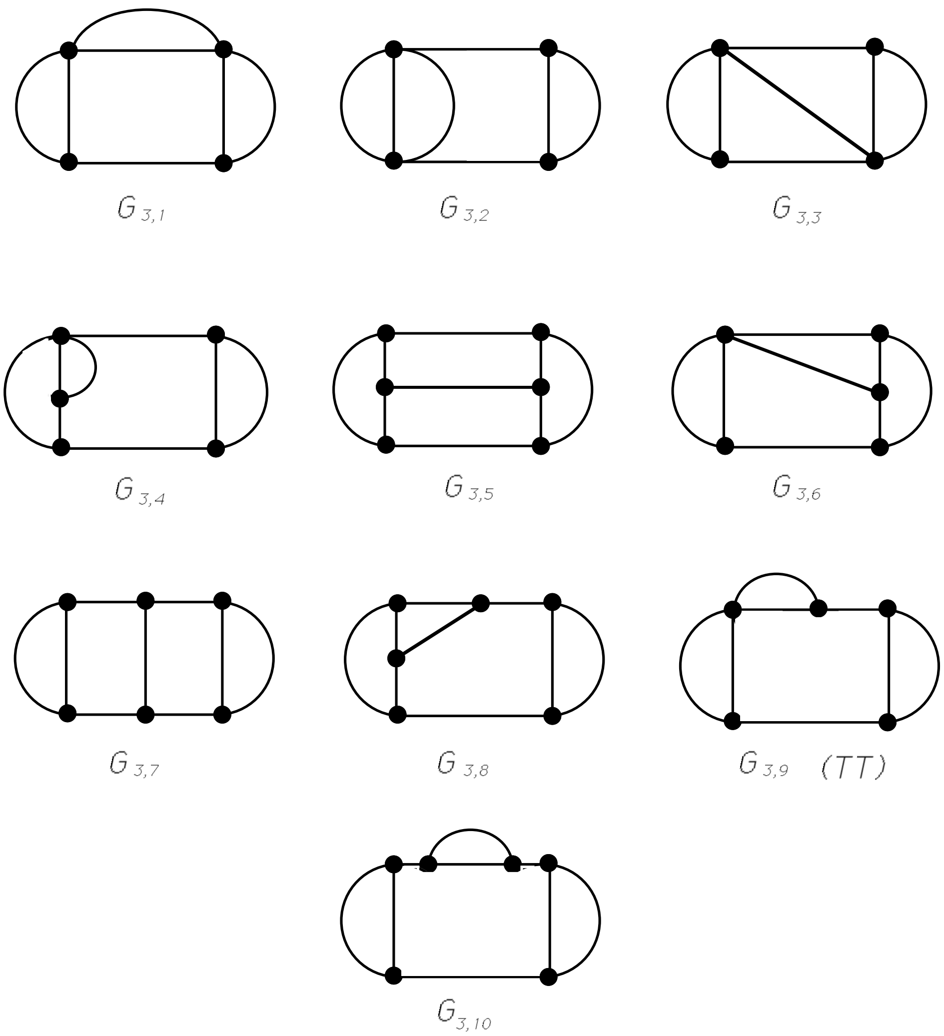

Figure 5: The ten open linear extensions of L_2
The 10th is a necklace, all other $\gamma_{avg} \geq 13/12$

$\frac{31}{24}$. The graph $G_{3,8}$ is homeomorphic to $G_{1,6}$ which has average genus $\frac{19}{16}$. The graph $G_{3,5}$ is homeomorphic to $G_{1,4}$ which has average genus $\frac{43}{32}$. We can also directly compute[5]

$$\gamma_{avg}(G_{3,1}) = \frac{11}{9} \qquad \gamma_{avg}(G_{3,3}) = \frac{5}{4}$$

$$\gamma_{avg}(G_{3,4}) = \frac{7}{6} \qquad \gamma_{avg}(G_{3,7}) = \frac{9}{8}$$

$$\gamma_{avg}(G_{3,9}) = \frac{13}{12}$$

Finally, the graph $G_{3,10}$ is a necklace of type $(3,0)$ which has average genus less than 1. However, suppose H is a 2-connected graph that contains a subgraph homeomorphic to $G_{3,10}$ and is not a necklace, then by Theorem 2.2, H can be built up from $G_{3,10}$ by adding open paths. Thus there must be a path P_i of H such that $P_0+P_1+\cdots+P_i$ contains a subgraph homeomorphic to one of the 9 graphs $G_{3,j}$, $j = 1, 2, \cdots, 9$ (note that all the graphs in Figure 5 are obtained by adding an open path to the necklace of type $(2,0)$). Thus we have the following:

Remark Let H be a 2-connected graph with no self-loops such that H is not a necklace, but that H contains a subgraph which is homeomorphic to the necklace $N_{(2,0)}$ of type $(2,0)$. Then the average genus of H is at least as large as the average genus of one of the graphs $G_{3,j}$, where $j = 1, 2, \cdots, 9$ in Figure 5.

Finally, let us consider the graph PM in Figure 2 (PM has average genus $\frac{5}{6}$). There are eleven non-homeomorphic graphs that can be obtained by adding an open path to PM. Figure 6 lists all these graphs. Call these graphs $G_{4,1}$, $G_{4,2}$, $\cdots$, $G_{4,11}$, respectively.

The graph $G_{4,1}$ is homeomorphic to $G_{2,1}$ which has average genus $\frac{31}{24}$. The graph $G_{4,5}$ is homeomorphic to $G_{3,3}$ which has average genus $\frac{5}{4}$. The graph $G_{4,6}$ is homeomorphic to $G_{3,1}$ which has average genus $\frac{11}{9}$. The graph $G_{4,10}$ is homeomorphic to $G_{3,4}$ which has average genus $\frac{7}{6}$, and the graph $G_{4,11}$ is homeomorphic to $G_{3,9}$ which is the graph TT and has average genus $\frac{13}{12}$. Moreover, the graphs $G_{4,4}$, $G_{4,7}$, $G_{4,8}$, and $G_{4,9}$ are homeomorphic to

[5] We will see that $13/12$ is the smallest real number > 1 that is the average genus of 2-connected graphs without self-loops. Thus we give the graph $G_{3,9}$ which reaches this average genus a special name: TT, for "thirteen-twelfths".

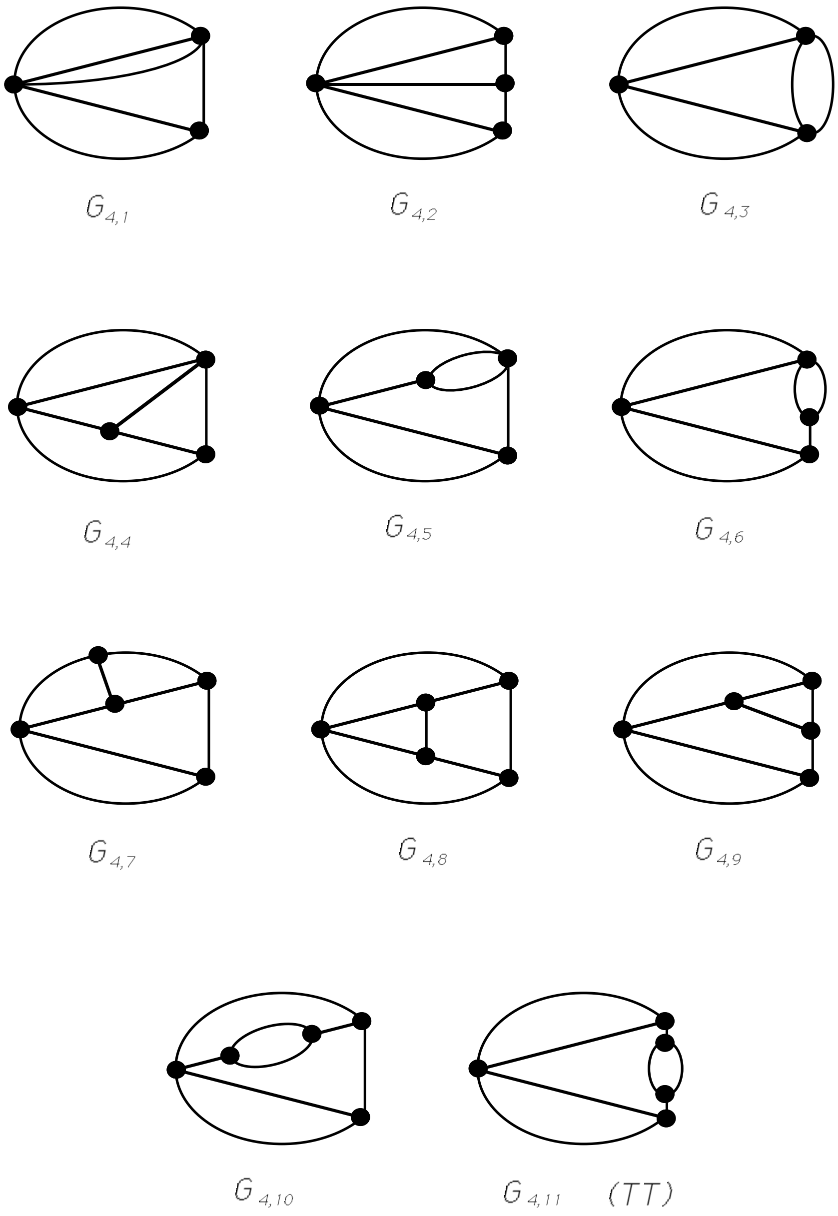

Figure 6: The eleven open linear extensions of PM
(all $\gamma_{avg} \geq 13/12$)

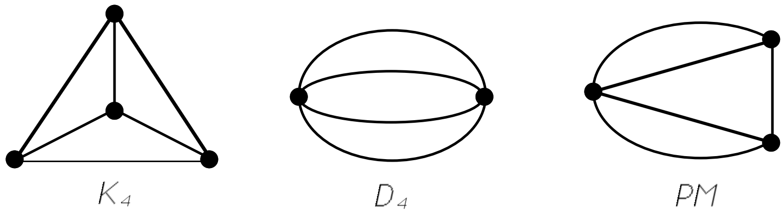

Figure 7: The loopless non-necklaces of average genus $< \frac{13}{12}$

K_4 plus an open path, so each of them is homeomorphic to one of the graphs in Figure 3. Finally, a direct computation gives us

$$\gamma_{avg}(G_{4,2}) = \frac{31}{24} \qquad\qquad \gamma_{avg}(G_{4,3}) = \frac{35}{27}$$

Summarize all these results, we obtain the following theorem.

Theorem 3.4 *Let G be a 2-connected graph with no self-loops such that the average genus of G is less than $\frac{13}{12}$. Then G must be either a necklace or one of the three graphs in Figure 7. In other words, a 2-connected non-necklace G without self-loops has average genus less than $\frac{13}{12}$ if and only if the length of the linear synthesis of G from a cycle is 2.*

Now suppose that G is a 2-connected graph that contains some self-loops, and that G is not a necklace. By Lemma 2.4, if we are only interested in the graphs of average genus less than or equal to 1, we can suppose that G is obtained by adding self-loops to a graph H which is either a necklace or one of the graphs in Figure 7. One thing we should point out is that since we are assuming that the graph H is the graph obtained by deleting self-loops from G, when we add back the self-loops, we should add them to the vertices or middle of edges of the graph H. In other words, a self-loop cannot be added to the middle of another self-loop. We will follow the same style we have done so far: starting with a loopless graph H which has average genus less than 1, we try to add self-loops in all possible ways, and stop when we find that the average genus of the graph exceeds 1.

Remark The average genus of the bouquet B_4 is 16/15. Thus, we need

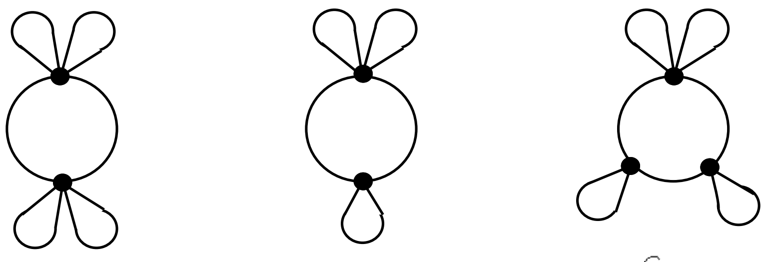

Figure 8: Three closed extensions of a cycle that are not necklaces
(The ones with average genus $\leq 28/27$)

not consider adding more than two self-loops at any vertex in what follows.

Let us start with a simple cycle C_1, which is the only graph in the class $2C_0$. Since G is not a necklace, there are two self-loops in G that are located at the same point v_1 on the cycle C_1. Therefore, G contains a subgraph that is homeomorphic to the bouquet B_3, which has average genus $\frac{2}{3}$. We consider all the possible cases as follows:

1. If there is another point v_2 on C_1 on which two other self-loops are located, then G contains a subgraph which is homeomorphic to the graph $C_{1,1}$ in Figure 8 which has average genus $\frac{11}{9}$.

2. If v_1 is the only point on C_1 on which there are more than one self-loop located. Then we add the third self-loop to a point v_2 on C_1 which is different from v_1, we get a graph which is homeomorphic to the graph $C_{1,2}$ in Figure 8 which has average genus $\frac{8}{9}$. If we further add a fourth self-loop to a third point v_3 on C_1, we get a graph which is homeomorphic to the graph $C_{1,3}$ in Figure 8 which has average genus $\frac{28}{27}$.

Using exactly the same idea, we can also work on the other loopless graphs with average genus less than 1. We just list all the results without detailed explanations, as follows:

- If we start with D_3, which is the only graph in class $2C_1$, then we can get 4 nonhomeomorphic 2-connected graphs with average genus less than or equal to 1. These are the graphs $D_{3,1}$, $D_{3,2}$, $D_{3,3}$ and $D_{3,4}$ in Figure 9. Here we have

$$\gamma_{avg}(D_{3,1}) = \frac{3}{4} \qquad \gamma_{avg}(D_{3,2}) = \frac{17}{18}$$

$$\gamma_{avg}(D_{3,3}) = \frac{8}{9} \qquad \gamma_{avg}(D_{3,4}) = 1$$

  Moreover, the graph $D_{3,5}$ is the only graph obtained in this way such that its average genus is $\frac{28}{27}$ which is the smallest value greater than 1.

- If we start with K_4, there are essentially two ways to add a self-loop, and the resulting graphs have average genus $\frac{19}{16}$ and $\frac{13}{12}$, respectively.

- If we start with D_4, there are again essentially two ways to add a self-loop, and the resulting graphs have average genus $\frac{6}{5}$ and $\frac{19}{18}$, respectively.

- If we start with PM, then the two graphs of smallest average genus obtained in this way are the graphs PM_1 and PM_2 in Figure 10. Here we have

$$\gamma_{avg}(PM_1) = 1 \qquad \gamma_{avg}(PM_2) = \frac{19}{18}$$

Remark Adding even one self-loop to K_4 or to D_4 drives the average genus up to at least 19/18.

Thus, we have exhausted all loopless non-necklace graphs with average genus less than 1. Now let us consider the case for necklaces. The cases for $N_{(0,0)}$, which is the simple cycle, and for $N_{(1,0)}$, which is the dipole D_3, have been discussed above.

If we start with $N_{(2,0)}$, the two graphs we obtain that have the smallest average genus are the graphs $N_{(2,0),1}$ and $N_{(2,0),2}$ in Figure 11. Here we have

$$\gamma_{avg}(N_{(2,0),1}) = 1 \qquad \gamma_{avg}(N_{(2,0),2}) = \frac{25}{24}$$

Let us consider $N_{(3,0)}$. It is routine to see that adding a self-loop to $N_{(3,0)}$ brings the average genus up to at least $\frac{25}{24}$. Now since all necklaces

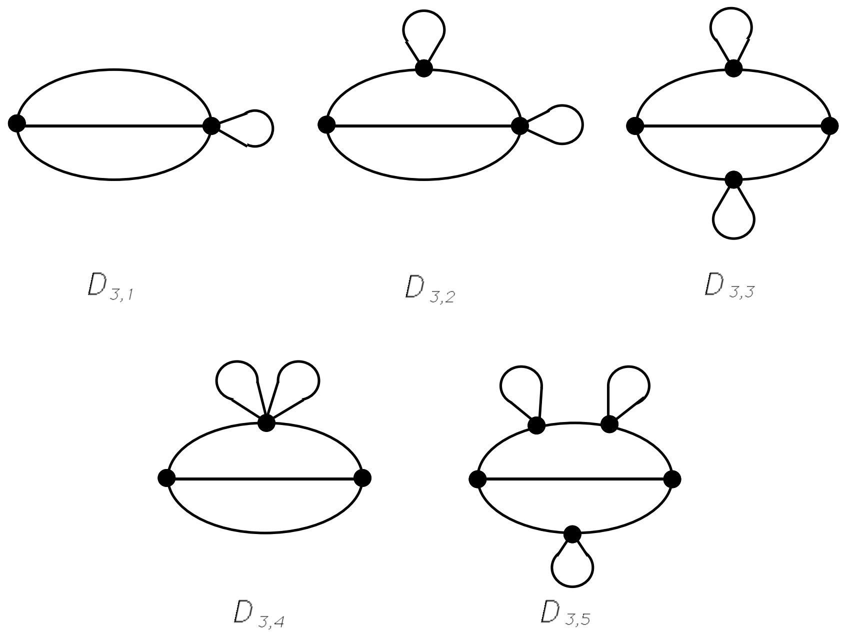

Figure 9: Five closed extensions of D_3 that are not necklaces
(The ones with average genus $\leq 28/27$)

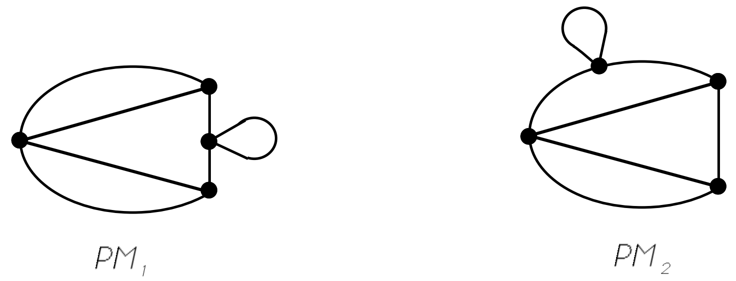

Figure 10: The two closed extensions of PM with smallest average genus

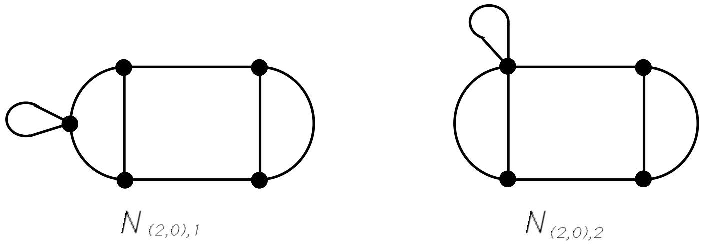

Figure 11: The two closed extensions of $N_{(2,0)}$ with smallest average genus

of type $(r, 0)$ with $r > 3$ have $N_{(3,0)}$ as a subgraph, thus adding a self-loop to $N_{(r,0)}$ such that the resulting graph is not a necklace makes the average genus of the resulting graph at least as large as $\frac{25}{24}$.

We can also start with a single vertex graph (the graph has one vertex and no edges) and add self-loops. Then we get a sequence of bouquets. However, we have only three bouquets that have average genus less than 1:

$$\gamma_{avg}(B_1) = 0 \qquad \gamma_{avg}(B_2) = \frac{1}{3} \qquad \gamma_{avg}(B_3) = \frac{2}{3}$$

and $\gamma_{avg}(B_4) = \frac{16}{15}$. We observe that B_1 and B_2 are both necklaces.

Now we are in a position of summarizing the results in this section. We list all 2-connected graphs, except for necklaces, of average genus less than 1.

Theorem 3.5 *There are exactly eight 2-connected non-necklaces with average genus less than 1.*

Proof. By the discussions above, a 2-connected non-necklace has average genus less than 1 if and only if it is homeomorphic to one of the graphs in Figure 12. □

Theorem 3.6 *There are exactly three 2-connected graphs with average genus equal to 1.*

Proof. The three graphs are the graph $D_{3,4}$ in Figure 9, the graph $N_{(2,0),1}$ in Figure 11, and the graph PM_1 in Figure 10. □

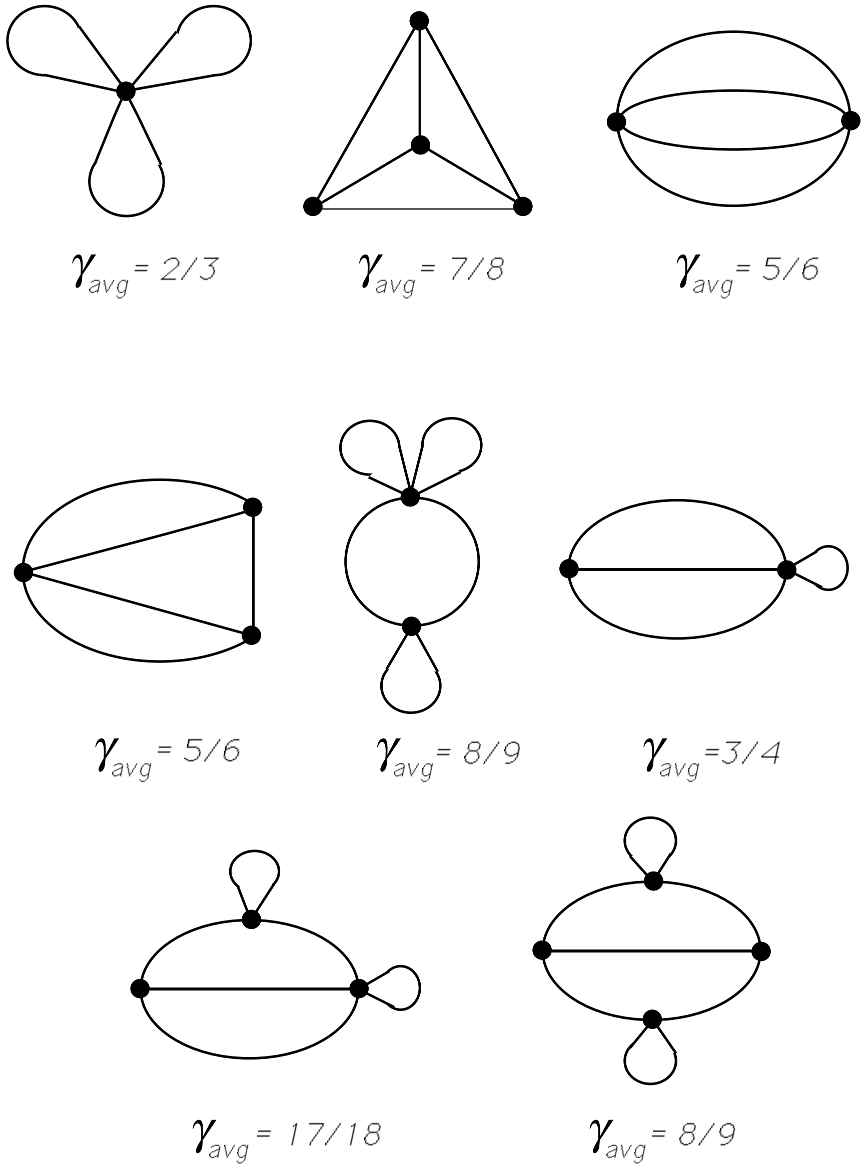

Figure 12: The 2-connected non-necklaces of average genus < 1

Corollary 3.7 *A real number r in the half-open interval $[0,1)$ is a value of average genus of 2-connected graphs if and only if r is of the form*

$$1 - \frac{2^{s-r}}{3^s}$$

where $s, r = 0, 1, \cdots$ are nonnegative integers.

PROOF. Suppose that G is a 2-connected graph with average genus less than 1. As we have observed, if G is a necklace, the average genus of G is of the form

$$1 - \frac{2^{s-r}}{3^s}$$

where $s, r = 0, 1, \cdots$ are nonnegative integers. If G is not a necklace, the average genus of G can only take the following values

$$0, \ \frac{1}{3}, \ \frac{1}{2}, \ \frac{2}{3}, \ \frac{3}{4}, \ \frac{5}{6}, \ \frac{7}{8}, \ \frac{8}{9}, \ \frac{17}{18}$$

However, all these numbers are also of the form $1 - \frac{2^{s-r}}{3^s}$. $\square$

Corollary 3.8 *The number 1 is not a lower limit point of values of average genus of 2-connected graphs.*

PROOF. As we have seen in the above analysis, the smallest real number greater than 1 that is a value of average genus of some 2-connected graph is $\frac{28}{27}$. $\square$

Corollary 3.9 *Given any real number $\epsilon > 0$, there are only finitely many 2-connected graphs with average genus $\leq 1 - \epsilon$.*

PROOF. By Corollary 3.7, in the interval $[0,1)$, only the real numbers of form $1 - \frac{2^{s-r}}{3^s}$ can be a values of average genus of 2-connected graphs, where s and r are nonnegative integers. Now there are only finitely many nonnegative integers r and s satisfying the inequality

$$1 - \frac{2^{s-r}}{3^s} \leq 1 - \epsilon$$

and for each fixed pair (r, s), there are only finitely many 2-connected graphs whose average genus is equal to $1 - \frac{2^{s-r}}{3^s}$. $\square$

Corollary 3.10 *The number 1 is the smallest limit point of the values of average genus of 2-connected graphs.*

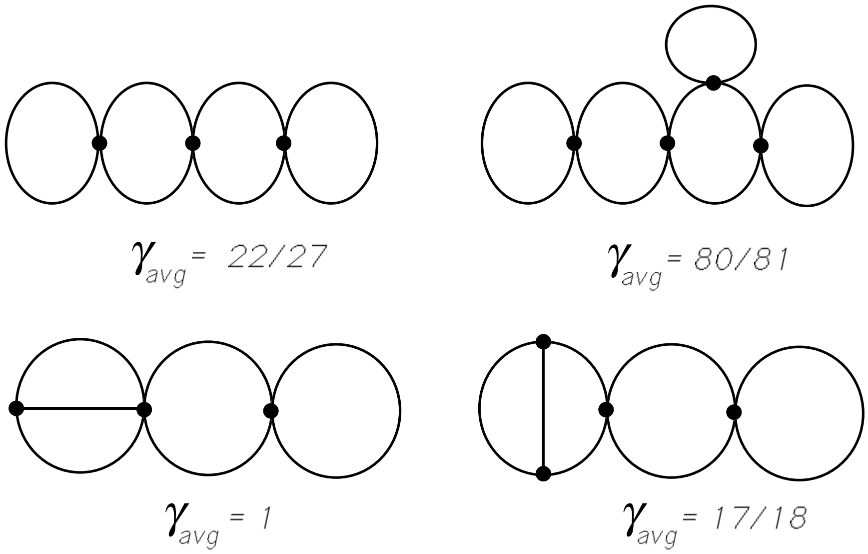

Figure 13: The four cutedge-free non-2-connected graphs of average genus ≤ 1

4 Other graphs with small average genus

Now we discuss the average genus of graphs that are not 2-connected. The graphs that are not 2-connected can be divided into two classes: those containing cutedges and those containing no cutedges.

Let $G = (V, E)$ be a connected graph and $e \in E$. The edge e is called a *cutedge of* G if its removal disconnects the graph G. We first look at the graphs containing no cutedges.

By Theorem 2.3, we can build up a connected graph without cutedges, by adding a sequence of paths, started with, say a simple cycle. Using an idea entirely similar to what we used in the last section, we can construct all cutedge-free graphs of average genus ≤ 1. Rather than write the details of the whole discussion and the lengthy computations, we summarize our results as follows:

Theorem 4.1 *Let G be a connected graph without cutedges, such that G is not 2-connected. Then G has average genus less than or equal to 1 if and only if G is homeomorphic to one of the four graphs in Figure 13.*

We close this section by discussing the average genus of graphs containing cutedges. There are certainly infinitely many nonhomeomorphic graphs with cutedges that have average genus less than 1. For example, all trees have average genus 0. Moreover, since "planting" a tree on a graph does not change the average genus of the graph, any graph G of average genus less than 1 plus any tree planted on it gives us a graph of average genus less than 1. For orientable imbeddings of graphs, there is another class of graphs that have average genus 0. They are called "cacti", and we define them as follows

Definition A *cactus* can be constructed in the following way: start with a tree T, then replace some of the vertices in T by simple cycles and connect the edges incident on each such vertex to the corresponding cycle in arbitrary way.

The following theorem was proved by Nordhaus, Ringeisen, Stewart and White.

Theorem 4.2 *([NRSW 1972]) Let G be a graph that is not a cactus. Then the maximum genus of G is greater than 0.*

Corollary 4.3 *The graph has average genus 0 if and only if it is a cactus.*

PROOF. This follows immediately from Theorem 4.2. □

A *bar-amalgamation* of the graphs G and H is obtained by joining a vertex of G to a vertex of H with a new edge.

Theorem 4.4 *The average genus of a bar-amalgamation of a graph G and a cactus equals the average genus of G.*

PROOF. A direct consequence of Corollary 4.3 and Theorem 5 of Gross and Furst [GrFu 1987]. □

Therefore, when we discuss the average genus of graphs, we can avoid cacti. We introduce the concept of a *cactus-free* graph, which is inductively defined as follows:

1. Every cutedge-free graph that is not a simple cycle is cactus-free.

2. The bar-amalgamation of two cactus-free graphs is cactus-free.

The intuitive idea of a cactus-free graph G is that when all the cutedges are deleted from G, none of the components of the resulting graph is a simple cycle or an isolated vertex.

Lemma 4.5 *The smallest four numbers that are values of average genus of cactus-free graphs are* $1/3$, $1/2$, $5/9$ *and* $2/3$.

PROOF. By Theorem 2.3, each cutedge-free graph has a linear synthesis from a cycle. Consider all possible linear syntheses of length ≤ 2. It is easy to see that the following four cutedge-free non-cycle graphs have the smallest average genus: B_2, D_3, $N_{(0,2)}$, and B_3, with average genus $1/3$, $1/2$, $5/9$, and $2/3$, respectively. If a cactus-free graph G is not cutedge-free, then it is a bar-amalgamation of two cactus-free graphs G_1 and G_2. The average genus of a bar-amalgamation equals the sum of average genera of the components [GrFu 1987]. If both G_1 and G_2 are homeomorphic to B_2, then G has average genus $2/3$. Otherwise, G will have average genus at least $1/3 + 1/2 = 5/6$. $\square$

Theorem 4.6 *Let G be a cactus-free graph with a cutedge such that the average genus of G is less than 1. Then G is homeomorphic to a bar-amalgamation of the bouquet B_2 with one of the following three graphs: B_2, D_3, and $N_{(0,2)}$.*

PROOF. Since G is not cutedge-free, G is a bar-amalgamation of two cactus-free graphs G_1 and G_2. Since $\gamma_{avg}(G) < 1$, at least one of G_1 and G_2 has average genus less than $1/2$ since

$$\gamma_{avg}(G) = \gamma_{avg}(G_1) + \gamma_{avg}(G_2)$$

Without loss of generality, suppose $\gamma_{avg}(G_1) < 1/2$, then by Lemma 4.5, G_1 is homeomorphic to B_2, and $\gamma_{avg}(G_1) = 1/3$. Now the average genus of G_2 must be less than $1 - \frac{1}{3} = \frac{2}{3}$. By Lemma 4.5 again, G_2 must be homeomorphic to one of B_2, D_3, and $N_{(0,2)}$. $\square$

We now restate a few theorems of Section 3 for more general classes of graphs.

Theorem 4.7 *A cactus-free graph G has average genus less than 1 if and only if either G is a necklace or G is homeomorphic to one of finitely many exceptions.*

Theorem 4.8 *The number 1 is not a lower limit point of the values of the average genus of cactus-free graphs.*

Theorem 4.9 *Given any positive real number $\epsilon > 0$, there are only finitely many nonhomeomorphic cactus-free graphs whose average genus is less than or equal to $1 - \epsilon$.*

5 A Kuratowski-type theorem for maximum genus 1

Kuratowski [Kuratowski 1930] proved that the minimum genus of a graph G is 0 if and only if G contains no subgraphs that are homeomorphic to either K_5 or $K_{3,3}$. Subsequently, any characterization of imbeddability by a list of obstructions has been called a "Kuratowski-type" result, as has any proof that there are only finitely many essential obstructions.

Glover, Huneke, and Wang [GlHuWa 1979] constructed a list of 103 independent obstructions to imbeddability in the projective plane, and Archdeacon [Archdeacon 1981] proved that the list is complete. The combination of these two results shows that the minimum crosscap number of a graph is at most 1 if and only if G contains no subgraphs that are homeomorphic to any graph in the list of the 103 graphs.

Using an entirely different approach involving graph minors, Robertson and Seymour [RoSe 1989] have proven Wagner's conjecture [Wagner 1937]: If a family of finite graphs is closed under minor reductions, then its obstruction set is finite. Wagner's conjecture implies that for any positive integer k, there is a finite list of graphs such that the minimum genus of a graph G is at most k if and only if G contains no subgraphs that are homeomorphic to any graph in the list.

A Kuratowski-type theorem was also derived for the maximum genus of graphs. Nordhaus, Ringeisen, Stewart and White [NRSW 1972] have proved that the maximum genus of a graph G is 0 if and only if G is a cactus.

The results in the previous sections of this paper have given us a way to characterize all graphs of maximum genus 1. It is clear that if a graph has average genus greater than 1, then its maximum genus must be greater than 1. Therefore, if a cactus-free graph has maximum genus 1, then it must be in the list of the graphs we have considered.

We start by considering those graphs in our list that are 2-connected and of average genus 1. Look at the graph $D_{3,4}$ in Figure 9, the graph $N_{(2,0),1}$ in

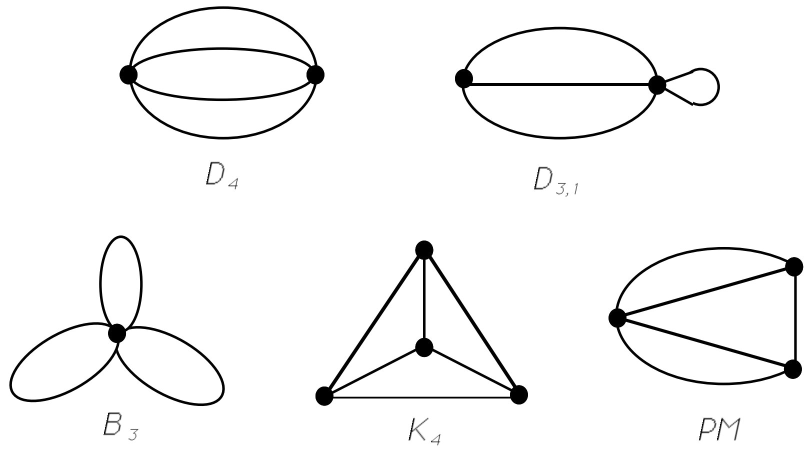

Figure 14: Cactus-free graphs with the maximum genus 1

Figure 11, and the graph PM_1 in Figure 10. These are the only 2-connected graphs of average genus 1. Since all these graphs are planar graphs, they must have genus 0 imbeddings. Therefore their average genus 1 implies that their maximum genus must be larger than 1.

Now we consider cutedge-free graphs. Look at Figure 13 for all cutedge-free graphs of average genus less or equal to 1 that are not 2-connected. Using Xuong's theorem [Xuong 1979], we easily check that all these graphs have the maximum genus greater than 1.

Finally, let us consider cactus-free graphs that are not cutedge-free. A cactus-free graph is not a cactus so its maximum genus is at least 1 (see Corollary 4.3). By the proof of Theorem 4.6, if all cutedges are removed from a cactus-free graph G that is not cutedge-free, the resulting graph contains at least two cutedge-free connected components that are not a simple cycle. Since a cutedge-free graph that is not a simple cycle has the maximum genus at least 1 and the maximum genus of a bar-amalgamation of two graphs is equal to the sum of the maximum genera of the two graphs [GrFu 1987], we conclude that the maximum genus of the graph G is at least 2.

Combining all these, we realize that if the maximum genus of a cactus-free graph G is 1, then G must be 2-connected and has average genus less than 1. Look at each of the graphs in Figure 12, use Xuong's theorem [Xuong 1979], we easily check that only the five graphs in Figure 14 are of the maximum genus 1. These graphs are exactly the set $2C_2$ of graphs that have length-2 linear synthesis from a cycle. We now conclude this section with the following theorem.

Theorem 5.1 *A cactus-free graph has maximum genus 1 if and only if it is homeomorphic to a necklace or obtainable by a length-2 linear synthesis from a cycle.*

6 Conclusions, open problems, and related results

We have presented a Kuratowski-type theorem for the average genus of graphs. Whitney's classical result [Whitney 1932] on edge deletion was used in the proof of the above result. Many interesting open problems are available for further research.

We have proved that with only eight exceptions, a 2-connected graph of average genus less than 1 must be a necklace. A necklace can be constructed by adding one edge to a cactus. From Figure 12, we find out that all those exceptional 2-connected graphs of average genus less than 1 can be constructed by adding at most two edges to a cactus. We can verify through Figure 13 that the four more exceptional cutedge-free graphs of average genus less than 1 can also be constructed by adding at most two edges to a cactus. Therefore, each cutedge-free graph of average genus less than 1 must be represented as a cactus plus at most two edges. However, there are also many graphs that are cacti plus two edges, but of average genus greater than 1. All the graphs in Figure 5 except $G_{3,9}$ are such examples. Is there a way to characterize these two different kinds of cutedge-free graphs ?

The techniques we adopted should be applicable to characterizing all graphs of average genus less than any fixed constant c, possibly after a very lengthy examination. We believe that is possible for small constants, such as $c = 2$, or 3. However, it is not practical when c is large. Is there a way to implement this method systematically?

As we have seen, the number 1 is an upper limit point of the set of values of average genus of necklaces. We have also proved that the number 1 is not a lower limit point of the set of values of average genus of cactus-free graphs. Elsewhere, we show that neither the set of values of average genus of 3-connected graphs nor of 2-connected simplicial graphs has any limit points [ChGr 1991b], and that the set of values of average genus of cutedge free graphs has no lower limit points [ChGr 1991d].

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